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SUPERSYMMETRY
IN
1-DIMENSIONAL SYSTEMS

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Introduction

Supersymmetry is one of the most important concepts introduced in Theoretical Physics in the last 30 years. Despite the fact that no experimental evidence for it has been found so far, intensive theoretical studies on supersymmetric theories have continued for all these years. It is therefore reasonable to wonder what can be the main motivations for such an amount of efforts.

The answer is twofold. Firstly Supersymmetry is very fascinating from the theoretical point of view, because it unifies fermions (i.e. matter) with bosons (i.e. interaction) both in flat space (*Supersymmetry*) and in curved space-time (*Supergravity*). Secondly, Supersymmetry is still a promising solution to many problems in particle Physics, as we will see in a while. The main goal of particle Physics in the last 50 years has been the unification of the four interactions of Nature. Many successful models have been constructed, but it is a matter of fact that many problems are still unsolved. In fact, while the Grand Unification Theories provide a satisfactory solution for the unification of electromagnetic, weak and strong forces, the problem of how to incorporate gravity in this scheme is still an open and intriguing question. One of the main motivations for supersymmetric theories is the hope that Supersymmetry may solve the non-renormalizability problem of quantum gravity. In fact, as is well known, gravity is not a renormalizable interaction and therefore it can not be combined with the other forces which, on the contrary, are renormalizable. An important feature of Supersymmetric Field Theories is that they exhibit a better ultraviolet behaviour than standard Quantum Field Theories. The reason is basically that fermions and bosons come in pairs and contribute with opposite sign to higher order corrections. Therefore, in order to insert gravity into the unification picture, the strategy is to make it supersymmetric (*Supergravity*) in order to obtain a renormalizable theory. This hope is still alive even if many models of supergravity have failed to reach this goal. However, many other promising models have been put forward, such as Superstring theories, M-theory, higher-dimensional models etc. The last word in this direction has not been said yet and therefore the hope is still alive that these new supersymmetry models be renormalizable or even finite.

Another problem for which Supersymmetry turns out to be useful is the so-called *hierarchy problem*, that is the huge disparity between the energy scale at which

the GUT symmetry is broken ($\sim 10^{15}$ GeV) and the energy scale characterizing the breaking of the electroweak symmetry (Higgs mass: $10^2 \div 10^3$ GeV). The advantage of a supersymmetric theory is that these two scales may be built into the tree level effective potential and then, thanks to Supersymmetry, there are some (non-renormalization) theorems ensuring that higher-order corrections do not destroy the hierarchy.

As we said at the beginning, the most fascinating feature of Supersymmetry is that it unifies fermions (i.e. matter) and bosons (i.e. interaction). However the realization of such a symmetry was proven to be impossible in the context of standard relativistic Quantum Field Theories by a number of no-go theorems which forbid a direct symmetry transformation between fields of different spin. Among these theorems, the most famous is the celebrated Coleman-Mandula theorem [13], which claims that — in the framework of relativistic field theories — all the charge operators whose eigenvalues represent internal symmetries (electric charge, isospin, hypercharge) must be Lorentz-invariant, and therefore must commute with the energy, the momentum and the angular momentum operators. This in turn means that irreducible multiplets of symmetry groups cannot contain particles with different mass or different spins. The first possibility to circumvent the Coleman-Mandula theorem was proposed by Gel'fand and Likhtman [28] and Sakita and Gervais [29] in 1971, followed by Akulov and Volkov [56] and then by Wess and Zumino [57]. The way out was to give up one of the hypotheses of Coleman and Mandula, that is the requirement that only the symmetries forming Lie groups with real parameters were admitted. The new idea was to introduce an algebra whose elements obey *anticommutation relations* or, equivalently, symmetry transformations with *Grassmannian parameters*. This is precisely the case of the Susy algebra. In this manner it became possible to collect in the same multiplet particles of different spins provided one introduces, for each particle, its supersymmetric partner with equal mass and different spin (for example the *photino*, the partner of the photon, with spin $\frac{1}{2}$ and the *selectron*, the partner of the electron, with spin 0).

Unfortunately, as the experiments have shown, no boson with the mass of an electron has been found and the same holds also for all the known particles of the Standard Model. This means that if Supersymmetry is a true symmetry of Nature, it must be broken at some energy scale. Therefore many models have been put forward, since the mid seventies, to provide satisfactory mechanisms to break Susy in a manner consistent with phenomenology. It was in this context that people tried to test the breaking mechanism on models simpler than the phenomenological ones. In particular Supersymmetry was first studied in a nonrelativistic framework by Witten [59] who invented a toy model called Supersymmetric Quantum Mechanics (Susy-QM).

It became clear soon that this model was interesting in its own right, not just as a toy model for testing field theory methods. For example, the introduction by

Witten [60] of a topological invariant (the Witten index) to study the breaking of Susy gave rise to a lot of interest in the geometrical and topological aspects of Susy-QM. Many properties of the Witten index were analyzed: for instance it was possible [4] to give a proof of the Atiyah-Singer index theorem by use of the supersymmetric representation of the index theorem.

In Atomic Physics, Kostelecký et al. [43] discussed the relationship between the physical spectra of different atoms and ions by use of Susy-QM. For example they suggested that the helium and hydrogen spectra come from Susy partner potentials.

In Nuclear Physics Supersymmetry was applied [38] to obtain relations between the spectra of even-even and neighbouring even-odd nuclei; it was shown [7] that the deep and shallow nucleus-nucleus potentials are supersymmetric partners.

In the general context of QM it was shown [32][14] that the factorization method of Schroedinger, Infeld and Hull [39] has its roots in a sort of SUSY-QM which one can build out of any QM model. SUSY-QM made its appearance also in the context of stochastic processes [48] where it was shown to be linked to the interplay between forward and backward Fokker-Planck dynamics [31].

Therefore SUSY-QM has turned out to be a model which makes its appearance in a large variety of areas in physics. It somewhat indicates a sort of *universality* of this 1-dimensional supersymmetry. However, a really universal supersymmetry is obtained in a model introduced some time ago [34] which is related to Classical Mechanics (CM) and not to Quantum Mechanics. At first sight this is very surprising because one could ask which are the fermionic variables in CM. The issue appears less “bizarre” if one remembers that in differential geometry some objects have been introduced, whose character is indeed anticommuting. They are the basis dx^i of the cotangent space T^*M to a given manifold M , and it is well known that the product between these objects is the *wedge*-product which is anticommuting. It can be shown that the geometric object which determines the evolution of a generic function of x^i and dx^i is the *Lie derivative* along the Hamiltonian flow. This object somehow treats at the same level both x and dx and exhibits a symmetry which exchanges these variables. This symmetry obviously turns a bosonic variable x into a fermionic variable dx and in this sense it is a (classical) supersymmetry. Moreover it is a *universal* supersymmetry, as we have already said above, because any classical evolution is described via a Lie derivative and this object always exhibits this 1-dimensional supersymmetry. This universality is something we consider very important because it may tell us that supersymmetry is a wider phenomenon than we previously thought. It is anyhow a phenomenon strictly intertwined with geometry and which may be difficult to detect at the pure phenomenological level of particle physics.

Originally the formalism we have just mentioned was born out of an attempt to provide a *path integral* formulation for Classical Mechanics [34], which turned

out to be the classical counterpart of the operatorial approach to CM pioneered by Koopman and Von Neumann [42]. This path integral, which we will indicate as CPI¹, exhibits a lot of universal symmetries, some of which were understood geometrically [34] thanks to the tools provided by the Cartan Calculus of symplectic spaces [1].

A symmetry which is present in the model, but not thoroughly analyzed geometrically is supersymmetry. In this thesis we will first do that and we will show that the universal supersymmetry of Classical Mechanics is strictly related to the geometrical concept of *equivariant cohomology*, which is a concept first introduced by Cartan [11] and developed by other people [54][8][9]. The strategy we will use is to make local the global Classical Susy present in the CPI-Lagrangian and then study the physical state conditions associated to this local invariance. The *physical* states turn out to be those belonging to the equivariant cohomological classes. The analysis we perform will also be useful as the starting point for the study of the geometry of the space of classical orbits.

In this thesis we shall also present other modifications of the original CPI formalism. Some of them will help us to make contact with structures (like the κ -symmetry of Siegel) found in many supersymmetric theories like the relativistic superparticles, strings and branes. Other modifications will allow to study structures typical of Classical Mechanics like the surfaces of constant energy.

Besides these extensions of the CPI, in this thesis we have also tried to find new universal symmetries not previously discovered in the literature [18]. One symmetry, whose generator will be indicated by \mathcal{Q}_s , has the effect of rescaling the overall action \tilde{S} which enters the CPI. It is a symmetry because the equations of motion are left invariant by a rescaling of \tilde{S} . However, it is a symmetry which cannot be implemented by canonical commutators on the phase space $\tilde{\mathcal{M}}$ of the CPI. The reason is that a rescaling of the action \tilde{S} is not a canonical transformation. However, its action is better understood if we make use of the concept of superspace. In fact, like any supersymmetric theory, the CPI formalism can be represented in a suitable superspace, which is composed by the time t and by two Grassmannian partners θ and $\bar{\theta}$. This superspace $(t, \theta, \bar{\theta})$ is like the base space for the target space $\tilde{\mathcal{M}}$ of our theory. The charge \mathcal{Q}_s performs a rescaling of $(\theta, \bar{\theta})$ while its action on $\tilde{\mathcal{M}}$ is non-canonical. This transformation seems to be the counterpart, at the level of the CPI, of a symmetry discovered before [33] in the context of the quantum path integral. That transformation, which we denote by MSA², is a transformation of the coordinates of the standard phase space \mathcal{M} and was shown to be a non-integrable one because it is *path dependent* (and not reducible

¹“CPI” is the acronym of “Classical Path Integral”.

²From “Mechanical Similarity Anomaly”. The name “Mechanical Similarity” was first introduced, even if in a different context, by Landau in Ref.[44] and the term “Anomaly” refers to the fact that it is broken in the passage from the classical to the quantum level.

to a diffeomorphism). It was also shown that at the quantum mechanical level this classical symmetry is anomalous because of the presence of \hbar . In the context of the CPI we shall show that also the transformation induced by \mathcal{Q}_S cannot be reduced to a diffeomorphism in the phase space $\widetilde{\mathcal{M}}$. The transformation which is closer to the \mathcal{Q}_S -rescaling is that known as *superconformal* transformation. In the last part of the thesis we shall present a model which exhibits a kind of superconformal invariance derived from the classical Susy of the CPI. Our hope is that this particular model may be a playground to tackle the more general problems of the transformations induced by \mathcal{Q}_S .

This thesis is organized as follows.

In Section 1 we shall introduce the formalism of the Path Integral for Classical Mechanics. We will show how to derive it starting from the Hamilton equations, and we will describe all the symmetries it possesses and their geometrical meaning by use of the Cartan Calculus.

In Section 2 we shall focus on the classical supersymmetry of the CPI and in particular we will analyze its geometrical meaning. We will modify the CPI-Lagrangian and we will obtain a gauge theory which exhibits the Classical Susy as a local symmetry. Next we will show how the physical Hilbert space is connected with the geometric concept of equivariant cohomology and with the geometry of the space of the classical trajectories. Besides the meaning of the classical supersymmetry, we shall discuss also another topic: the problem of constraining the CPI-formalism on the hypersurfaces of the phase space \mathcal{M} characterized by a fixed value E of the energy. In other words we will insert in the CPI the constraint $H(\varphi) - E = 0$ and we will analyze the associate gauge theory. We will see that the original $N = 2$ classical supersymmetry reduces to a $N = 1$.

In Section 3 we will continue the analysis of the symmetries of the CPI. We shall show that, if we make local the symmetries generated by the operators associated to the covariant derivatives of the SUSY, we obtain a local (super)symmetry which is very similar to the κ -symmetry of Siegel.

In Section 4 we will introduce the symmetry generated by \mathcal{Q}_S and we will analyze it both at the level of superspace and at the level of the phase space $\widetilde{\mathcal{M}}$. We shall show that this symmetry is somewhat similar to the MSA-transformation introduced above, which is also analyzed in detail.

In Section 5 we will introduce the superconformal invariance in the CPI. To do that, we will build the Lie derivative of a model introduced long time ago by De Alfaro, Fubini and Furlan [15] which exhibits a particular kind of conformal invariance. Since the Lie derivative is automatically supersymmetric, we will get a model which combines the invariance under conformal symmetries with the invariance under susy: in other words we will obtain the invariance under the so-called superconformal transformations. To obtain all the symmetry group at the level of $\widetilde{\mathcal{M}}$, we will build the Lie derivatives associated to all the conformal generators of

the theory. Moreover we will construct also the “square root” of the charges associated to each of the previous generators and we will calculate the algebra realized by all these operators. They close on a new kind of superconformal algebra.

At the end some conclusions summarize the work done and indicate the open problems ahead. Few appendices contain some detailed calculations omitted in the previous sections.

1. The Path Integral for Classical Mechanics

In this chapter we shall describe the formalism of the Classical Path Integral (CPI) which was originally developed in [34], where the interested reader can find all the details omitted in the present account. The idea originated from the fact that whenever a theory has an operatorial formulation, it must also possess a corresponding path integral. Now Classical Mechanics (CM) does have an operatorial formulation which was proposed long ago by Koopman and von-Neumann [42]. This operatorial approach describes the time-evolution of phase-space distributions by means of the *Liouville* operator $\hat{L} = -\omega^{ab}\partial_b H\partial_a$ where ω^{ab} is the symplectic matrix. Therefore, it is reasonable to look for the corresponding path integral formalism and the strategy to follow is described in the next section.

1.1 The Classical Kernel

Classical Hamiltonian Mechanics describes physical systems in a $2n$ -dimensional phase space \mathcal{M} , whose coordinates we will denote by φ^a ($a = 1, \dots, 2n$), i.e.: $\varphi^a = (q^1 \dots q^n, p^1 \dots p^n)$. If we call $H(\varphi)$ the Hamiltonian of the system, the equations of motion (i.e. the Hamilton equations) are:

$$\dot{\varphi}^a = \omega^{ab} \frac{\partial H}{\partial \varphi^b} \equiv \omega^{ab} \partial_b H(\varphi) \quad \omega^{ab} = \text{symplectic matrix.} \quad (1.1.1)$$

We can define the *classical* probability $K_{cl}(f|i)$ to reach a final configuration φ_f at time t_f given an initial configuration φ_i at time t_i in the following way:

$$K_{cl}(f|i) = \delta(\varphi_f^a - \phi_{cl}^a(t_f|\varphi_i, t_i)) \quad (1.1.2)$$

where $\phi_{cl}^a(t_f|\varphi_i, t_i)$ is the classical trajectory (that is the solution of the Hamilton equations) having φ_i as initial condition at time t_i . Since $K_{cl}(f|i)$ is a classical

probability, we can rewrite it as a sum over all the possible intermediate configurations:

$$\begin{aligned}
K_{cl}(f|i) &= \sum_{k_i} K_{cl}(f|k_{N-1}) K_{cl}(k_{N-1}|k_{N-2}) \cdot \dots \cdot K_{cl}(k_1|i) \\
&= \prod_{j=1}^N \int d^{2n} \varphi_j \delta^{(2n)}[\varphi_j^a - \phi_{cl}^a(t_j|\varphi_{j-1}, t_{j-1})] \\
&\xrightarrow{N \rightarrow \infty} \int \mathcal{D}\varphi \tilde{\delta}[\varphi^a(t) - \phi_{cl}^a(t)]
\end{aligned} \tag{1.1.3}$$

where in the first equality k_i denotes formally an intermediate configuration φ_{k_i} between φ_i and φ_f and the symbol $\tilde{\delta}(\dots)$ represents a *functional* Dirac delta, that is a product of an infinite number of Dirac deltas, each one referring to a different time t along the classical trajectory. The last formula in (1.1.3) is already a path integral but it does not have the usual form (the integral of the exponential of an action) we are used to see in Quantum Mechanics. Anyway we can give it that form if we rewrite the functional Dirac delta as:

$$\tilde{\delta}[\varphi - \varphi_{cl}] = \tilde{\delta}[\dot{\varphi}^a - \omega^{ab} \partial_b H] \det[\delta_b^a \partial_t - \omega^{ac} \partial_c \partial_b H] \tag{1.1.4}$$

where we have used the functional analog of the relation $\delta[f(x)] = \frac{\delta[x-x_i]}{\left| \frac{\partial f}{\partial x} \right|_{x_i}}$. Then

(see Refs.[34] for the details) we can exponentiate both the two terms of the RHS of Eq.(1.1.4): the first via a Lagrange multiplier λ_a and the second (the determinant) by use of two families of Grassmannian variables (c^a, \bar{c}_a) , as one usually does in the Faddeev-Popov technique in gauge theories [37]. We finally obtain the following expression:

$$K_{cl}(f|i) = \int_{\varphi_i, t_i}^{\varphi_f, t_f} \mathcal{D}\varphi^a \mathcal{D}\lambda_a \mathcal{D}c^a \mathcal{D}\bar{c}_a \exp \left[i \int dt \tilde{\mathcal{L}} \right] \tag{1.1.5}$$

where $\tilde{\mathcal{L}}$ is the Lagrangian characterizing the CPI:

$$\tilde{\mathcal{L}} = \lambda_a [\dot{\varphi}^a - \omega^{ab} \partial_b H] + i \bar{c}_a [\delta_b^a \partial_t - \omega^{ac} \partial_c \partial_b H] c^b. \tag{1.1.6}$$

From (1.1.5) we can pass to the classical generating functional Z_{CM}

$$Z_{CM}[J_\varphi, J_\lambda, J_c, J_{\bar{c}}] = \int \mathcal{D}\varphi^a \mathcal{D}\lambda_a \mathcal{D}c^a \mathcal{D}\bar{c}_a \exp \left[i \int dt (\tilde{\mathcal{L}} + J_\varphi \varphi + J_\lambda \lambda + J_c c + J_{\bar{c}} \bar{c}) \right], \tag{1.1.7}$$

where $J_\varphi, J_\lambda, J_c, J_{\bar{c}}$ are the currents associated to the $8n$ variables $(\varphi^a, \lambda_a, c^a, \bar{c}_a)$ which characterize the new *enlarged* phase space, which from now on we will denote by $\widetilde{\mathcal{M}}$. We will come back in a while on these things: for our purposes here, the

important thing is the form of the Lagrangian $\widetilde{\mathcal{L}}$ appearing in (1.1.6): we can easily Legendre-transform it obtaining the corresponding Hamiltonian:

$$\widetilde{\mathcal{H}} = \lambda_a \omega^{ab} \partial_b H + i \bar{c}_a \omega^{ac} (\partial_c \partial_b H) c^b. \quad (1.1.8)$$

The Lagrangian $\widetilde{\mathcal{L}}$ and the Hamiltonian $\widetilde{\mathcal{H}}$ will be extensively used in the following sections, where we will analyze some nice geometrical features they possess.

1.2 The enlarged phase space $\widetilde{\mathcal{M}}$

In the previous section we have seen that the phase space which characterizes the Classical Path Integral is larger than the original one (whose variables are φ^a), and is composed by the following $8n$ variables: $(\varphi^a, c^a, \lambda_a, \bar{c}_a)$. This phase space is endowed with both a symplectic and a commutator structure. The first one is defined as:

$$\{\varphi^a, \lambda_b\}_{EPB} = \delta_b^a; \quad \{c^a, \bar{c}_b\}_{EPB} = -i \delta_b^a \quad (\text{all others are zero}) \quad (1.2.1)$$

where the subscript “*EPB*” means *Extended Poisson Brackets* and the name has been chosen to emphasize the difference with the usual Poisson Brackets. The EPB allow to express the equations of motion³ of the $(\varphi^a, c^a, \lambda_a, \bar{c}_a)$ variables in a Hamiltonian form (where $\widetilde{\mathcal{H}}$ is the *extended* Hamiltonian). The second structure mentioned above is based on the definition of commutator given by Feynman in the quantum case: given a path integral characterized by the generating functional Z , we define the commutator between two functions O_1 and O_2 as the following (graded) expectation value:

$$\langle [O_1(t), O_2(t)] \rangle_Z \equiv \lim_{\epsilon \rightarrow 0} \langle O_1(t + \epsilon) O_2(t) \pm O_2(t + \epsilon) O_1(t) \rangle_Z. \quad (1.2.2)$$

If we use the same strategy with the Z_{CM} in (1.1.7), we get that the only commutators different from zero are:

$$\langle [\varphi^a, \lambda_b] \rangle_Z = i \delta_b^a; \quad \langle [c^a, \bar{c}_b] \rangle_Z = \delta_b^a \quad (1.2.3)$$

which are clearly isomorphic to the *EPB* in (1.2.1). The information we get either from (1.2.1) or from (1.2.3) is that the two pairs (φ, λ) and (c, \bar{c}) are couples of conjugate variables. In the following we choose to consider (φ, c) as forming the *enlarged configuration space* while we consider (λ, \bar{c}) as the conjugate momenta. This interpretation allows us to realize the commutators (1.2.3) via the following operatorial representations:

$$\widehat{\varphi^a} \rho(\varphi, c) = \varphi^a \cdot \rho(\varphi, c); \quad \widehat{\lambda_a} \rho(\varphi, c) = -i \frac{\partial}{\partial \varphi^a} \rho(\varphi, c); \quad (1.2.4)$$

$$\widehat{c^a} \rho(\varphi, c) = c^a \cdot \rho(\varphi, c); \quad \widehat{\bar{c}_a} \rho(\varphi, c) = \frac{\partial}{\partial c^a} \rho(\varphi, c); \quad (1.2.5)$$

³Obtained from the Euler equations associated to the Lagrangian $\widetilde{\mathcal{L}}$.

where $\rho(\varphi, c)$ is the analog of the wave function in the quantum case and, as we will see in a while, represents a generalization of the Koopman-Von Neumann wavefunction. If we substitute the differential representations (1.2.4) and (1.2.5) in the Hamiltonian $\widehat{\mathcal{H}}$ in (1.1.8), we get that $\rho(\varphi, c)$ evolves according to the following equation:

$$\widehat{\mathcal{H}}\rho(\varphi, c; t) = i \frac{\partial}{\partial t} \rho(\varphi, c; t). \quad (1.2.6)$$

where $\widehat{\mathcal{H}}$ is:

$$\widehat{\mathcal{H}} \equiv -i\omega^{ab}\partial_b H \frac{\partial}{\partial \varphi^a} - i\omega^{ab}\partial_b \partial_d H c^d \frac{\partial}{\partial c^a}. \quad (1.2.7)$$

Because of the Grassmannian character of the c 's, we can expand $\rho(\varphi, c; t)$ as:

$$\begin{aligned} \rho(\varphi, c; t) = & \rho_0(\varphi; t) + \rho_a(\varphi; t)c^a + \frac{1}{2}\rho_{ab}(\varphi; t)c^a c^b + \\ & + \dots + \frac{1}{(2n)!}\rho_{ab\dots d}(\varphi; t)c^a c^b \dots c^d. \end{aligned} \quad (1.2.8)$$

Now we can substitute Eq.(1.2.8) in (1.2.6) and equate separately all the terms with the same number of c 's. The first term in this expansion, $\rho_0(\varphi)$, turns out to be the solution of the Liouville equation

$$\frac{\partial}{\partial t} \rho_0 = -\omega^{ab}\partial_b H \partial_a \rho_0, \quad (1.2.9)$$

because the bosonic part of $\widehat{\mathcal{H}}$ is precisely (apart from a factor i) the Liouville operator

$$\widehat{\mathcal{H}}_B \equiv -i\omega^{ab}\partial_b H \partial_a. \quad (1.2.10)$$

Thus we see that we recover the Koopman-Von Neumann formalism as the zero-ghost sector of the CPI-formalism. The next step is then to understand the remaining (i.e. with a non-zero number of ghosts c^a) terms. To do that we must give the ghosts c^a a physical meaning and the best strategy is focusing on the equations of motion of these variables:

$$\partial_t c^a - \omega^{ab}\partial_b \partial_d H c^d = 0. \quad (1.2.11)$$

This is precisely the equation of motion of the Jacobi fields⁴ $\delta\varphi^a$. This leads us to an interpretation which may seem a little heuristic at first sight, but which will be very illuminating in the sequel: we interpret the ghosts c^a as the basis $d\varphi^a$ of the cotangent space $T^*\mathcal{M}$. According to this prescription the anticommuting

⁴The Jacobi fields are also called “first variations”. They describe the evolution in time of the difference between two close classical trajectories in the phase space \mathcal{M} . See Ref.[50] for further details.

product between the ghosts becomes the \wedge -product between the 1-forms $d\varphi^a$ and the function $\rho(\varphi, c; t)$ in (1.2.8) becomes

$$\rho(\varphi, c; t) \longrightarrow \rho(\varphi; t) = \rho_0(\varphi; t) + \rho_a(\varphi; t)d\varphi^a + \frac{1}{2}\rho_{ab}(\varphi; t)d\varphi^a \wedge d\varphi^b + \dots \quad (1.2.12)$$

which is an inhomogeneous differential form on the phase space \mathcal{M} . As we will see better in the following section, the Hamiltonian $\widetilde{\mathcal{H}}$ in (1.2.7) can then be interpreted as the *Lie-derivative* along the Hamiltonian vector field $h = \omega^{ab}\partial_b H \partial_a$, and the Schrödinger-type equation (1.2.6) describes the time evolution (induced by the Hamilton equations) of a generic inhomogeneous differential form.

We conclude this section with few words about the geometry of the enlarged phase space $\widetilde{\mathcal{M}}$. We have just seen that the ghosts c^a evolve as the basis of the cotangent space $T^*\mathcal{M}$, but they also transform as the components of the tangent vectors: $V^a(\varphi)\frac{\partial}{\partial\varphi^a}$. The space (φ^a, c^a) , which we have simply denoted by *enlarged configuration space* is called more properly in Ref.[51] the *reversed-parity* tangent bundle and it is indicated as $\Pi T\mathcal{M}$. The “*reversed-parity*” specification is because the c^a are Grassmannian variables. Since the (λ_a, \bar{c}_a) are the “momenta” of the previous variables (see Eq.(1.2.3)) we conclude that the $8n$ variables $(\varphi^a, c^a, \lambda_a, \bar{c}_a)$ span the cotangent bundle to the reversed-parity tangent bundle: $T^*(\Pi T\mathcal{M})$; in other words we can write $\widetilde{\mathcal{M}} = T^*(\Pi T\mathcal{M})$. So $\widetilde{\mathcal{M}}$ is a cotangent bundle and this is the reason why it has a Poisson structure. For more details about this we refer the interested reader to Ref.[36].

1.3 Symmetries of the CPI and the Cartan calculus

In this section we analyze the symmetries of the Hamiltonian $\widetilde{\mathcal{H}}$ and we focus in particular on their geometrical meaning. In fact the Hamiltonian $\widetilde{\mathcal{H}}$ is invariant under some *universal global* symmetries: *universal* because the transformations which leave \mathcal{H} invariant do not depend on the particular form of the original Hamiltonian $H(\varphi)$, and therefore are symmetries of *all* the classical systems; *global*, on the other side, is used in opposition to *local*, because the symmetries we are going to describe are not gauge symmetries. We will gauge some of these symmetries in the next chapter and we will see how we can gain an insight into their meaning; anyway for the moment no gauge invariance is present.

We can divide the generators (through the commutators in (1.2.3)) of these

symmetries in the following two groups:

Geometrical Symmetries

$$Q_{BRS} \equiv i c^a \lambda_a \quad (1.3.1)$$

$$\overline{Q}_{BRS} \equiv i \overline{c}_a \omega^{ab} \lambda_b \quad (1.3.2)$$

$$Q_g \equiv c^a \overline{c}_a \quad (1.3.3)$$

$$K \equiv \frac{1}{2} \omega_{ab} c^a c^b \quad (1.3.4)$$

$$\overline{K} \equiv \frac{1}{2} \omega^{ab} \overline{c}_a \overline{c}_b \quad (1.3.5)$$

Dynamical Symmetries

$$N_H \equiv c^a \partial_a H \quad (1.3.6)$$

$$\overline{N}_H \equiv \overline{c}_a \omega^{ab} \partial_b H \quad (1.3.7)$$

$$Q_H \equiv Q_{BRS} - \beta N_H = c^a (i \lambda_a - \beta \partial_a H) \quad (1.3.8)$$

$$\overline{Q}_H \equiv \overline{Q}_{BRS} + \beta \overline{N}_H = \overline{c}_a \omega^{ab} (i \lambda_b + \beta \partial_b H) \quad (1.3.9)$$

(β =dimensional parameter)

We called the first group “Geometrical” because the functional form of all its components does not contain $H(\varphi)$: we will see in a while that they correspond to some precise operations in differential geometry. On the other side, the second group is named “Dynamical” because its components depend formally on $H(\varphi)$; nevertheless they remain *universal*, because their form is the same for any classical system. Consider the first group: it is not difficult to check that the last three charges (Q_g, K, \overline{K}) realize the algebra $\mathfrak{sp}(2)$ while the first two ($Q_{BRS}, \overline{Q}_{BRS}$) provide the inhomogeneous part (this will be clearer in the following section) in such a way that the overall algebra of the first set of charges is $\mathfrak{is}\mathfrak{p}(2)$.

About the second group the most interesting charges are the last two, namely Q_H and \overline{Q}_H . In fact they are both nilpotent, they commute with $\tilde{\mathcal{H}}$ and their (graded) commutator is:

$$[Q_H, \overline{Q}_H] = 2i\beta \tilde{\mathcal{H}}; \quad (1.3.10)$$

this means that we have an $N = 2$ real (i.e. $N = 1$ complex) non-relativistic supersymmetry. This is precisely the classical supersymmetry we have mentioned in the Introduction.

In order to shed some light on the geometrical meaning of the charges defined in (1.3.1)-(1.3.9), we can exploit the correspondence $c^a \longleftrightarrow d\varphi^a$ introduced in the previous section together with the differential representations of $(\varphi^a, c^a, \lambda_a, \overline{c}_a)$ given in Eqs.(1.2.4)(1.2.5). For example it is not difficult to see

that:

$$\widehat{Q_{BRS}} = i\hat{c}^a\hat{\lambda}_a \quad \longleftrightarrow \quad d = d\varphi^a\partial_a \quad (\text{exterior differential}) \quad (1.3.11)$$

$$\hat{K} = \frac{1}{2}\omega_{ab}\hat{c}^a\hat{c}^b \quad \longleftrightarrow \quad \omega = \frac{1}{2}\omega_{ab}d\varphi^a \wedge d\varphi^b \quad (\text{symplectic 2-form}) \quad (1.3.12)$$

$$\widehat{N_H} = \hat{c}_a\omega^{ab}\partial_b H(\hat{\varphi}) \quad \longleftrightarrow \quad \iota_h \quad (\text{interior contraction w.r.t. } h) \quad (1.3.13)$$

In particular, since the Hamiltonian $\tilde{\mathcal{H}}$ can be written as:

$$\tilde{\mathcal{H}} = -i[Q_{BRS}, \overline{N_H}] \quad (1.3.14)$$

we have the following interpretation for $\tilde{\mathcal{H}}$:

$$i\tilde{\mathcal{H}} = [Q_{BRS}, \overline{N_H}] \longleftrightarrow d\iota_h + \iota_h d = \mathcal{L}_h \quad (1.3.15)$$

where \mathcal{L}_h is the *Lie-derivative* along the Hamiltonian vector field h , as we had anticipated in the previous section⁵.

We end this section giving a more rigorous formalization of the correspondence between the charges in \mathcal{M} and the Cartan Calculus [11][1].

First of all, since under the Hamiltonian evolution c^a transforms as the basis of 1-forms $d\varphi^a$ and \bar{c}_a transforms as the basis of vector fields⁶, (see Eqs.(1.2.4)-(1.2.5)), we can build the following map, called “hat” map \wedge :

$$\alpha = \alpha_a d\varphi^a \quad \xrightarrow{\wedge} \quad \hat{\alpha} \equiv \alpha_a c^a \quad (1.3.16)$$

$$V = V^a \partial_a \quad \xrightarrow{\wedge} \quad \hat{V} \equiv V^a \bar{c}_a \quad (1.3.17)$$

It is actually a much more general map between forms α , antisymmetric tensors V and functions of φ, c, \bar{c} :

$$F^{(p)} = \frac{1}{p!} F_{a_1 \dots a_p} d\varphi^{a_1} \wedge \dots \wedge d\varphi^{a_p} \quad \xrightarrow{\wedge} \quad \hat{F}^{(p)} \equiv \frac{1}{p!} F_{a_1 \dots a_p} c^{a_1} \dots c^{a_p} \quad (1.3.18)$$

$$V^{(p)} = \frac{1}{p!} V^{a_1 \dots a_p} \partial_{a_1} \wedge \dots \wedge \partial_{a_p} \quad \xrightarrow{\wedge} \quad \hat{V} \equiv \frac{1}{p!} V^{a_1 \dots a_p} \bar{c}_{a_1} \dots \bar{c}_{a_p} \quad (1.3.19)$$

Once the correspondence (1.3.16)-(1.3.19) is established, we can easily find out what corresponds to the various Cartan operations like the exterior derivative

⁵It had already been noted in Ref.[58] that the Lie-derivative has a supersymmetric structure. What was missing there was the detailed Cartan Calculus we present here.

⁶Note that λ_a does not seem to transform as a vector field even if it can be interpreted as $\frac{\partial}{\partial \varphi^a}$. The explanation of this fact is given in the second paper of Ref.[36].

d of a form, the interior contraction ι_V between a vector field V and a form F and the multiplication of a form by its form number [34] :

$$dF^{(p)} \xrightarrow{\quad} [Q_{BRS}, \widehat{F}^{(p)}] \quad (1.3.20)$$

$$\iota_V F^{(p)} \xrightarrow{\quad} [\widehat{V}, \widehat{F}^{(p)}] \quad (1.3.21)$$

$$pF^{(p)} \xrightarrow{\quad} [Q_g, \widehat{F}^{(p)}] \quad (1.3.22)$$

where Q_{BRS} , Q_g are the charges in (1.3.1)-(1.3.3). In the same manner we can translate in our language the usual mapping [1] between vector fields V and forms V^\flat realized by the symplectic 2-form $\omega(V, \cdot) \equiv V^\flat$, or the inverse operation of building a vector field α^\sharp out of a form: $\alpha = (\alpha^\sharp)^\flat$. These operations can be translated in our formalism as follows:

$$V^\flat \xrightarrow{\quad} [K, \widehat{V}] \quad (1.3.23)$$

$$\alpha^\sharp \xrightarrow{\quad} [\overline{K}, \widehat{\alpha}] \quad (1.3.24)$$

where again K, \overline{K} are the charges (1.3.4) and (1.3.5) respectively. We can also translate the standard operation of building a Hamiltonian vector field, indicated as $(df)^\sharp$, out of a function $f(\varphi)$, and also the Poisson Brackets between two functions f and g :

$$(df)^\sharp \xrightarrow{\quad} [\overline{Q}_{BRS}, f] \quad (1.3.25)$$

$$\{f, g\}_{PB} = df[(dg)^\sharp] \xrightarrow{\quad} [[f, Q_{BRS}], \overline{K}], [[[g, Q_{BRS}], \overline{K}], K]] \quad (1.3.26)$$

Finally we can translate the concept of Lie derivative \mathcal{L}_V along a generic vector field V ; from the definition

$$\mathcal{L}_V = d\iota_V + \iota_V d \quad (1.3.27)$$

it is easy to prove that:

$$\mathcal{L}_V F^{(p)} \xrightarrow{\quad} i[\widetilde{\mathcal{H}}_V, \widehat{F}^{(p)}] \quad (1.3.28)$$

where $\widetilde{\mathcal{H}}_V = \lambda_a V^a + i\overline{c}_a \partial_b V^a c^b$. Note that, for $V^a = \omega^{ab} \partial_b H$, the $\widetilde{\mathcal{H}}_V$ becomes the $\widetilde{\mathcal{H}}$ of (1.1.8) and (1.2.7).

1.4 The CPI-formalism in superspace

In the previous section we have seen that the formalism of the Classical Path Integral involves a universal $N = 2$ supersymmetry. It is very interesting to give a superspace formulation of the model because we can see that in this context there is a nice correspondence between the classical and the quantum domain of the same physical system.

First of all we must remember that we are dealing with a *non relativistic* supersymmetry and therefore our base space t enlarges to a superspace $(t, \theta, \overline{\theta})$ where θ

and $\bar{\theta}$ are two (scalar) Grassmannian partners of the time t . It is then possible to introduce a superfield which collects all the $8n$ -variables of $\widetilde{\mathcal{M}}$:

$$\Phi^a(t, \theta, \bar{\theta}) = \varphi^a(t) + \theta c^a(t) + \bar{\theta} \omega^{ab} \bar{c}_b(t) + i\bar{\theta}\theta \omega^{ab} \lambda_b(t). \quad (1.4.1)$$

The reason for the i in the last term is due to the choice of reality of the ghosts (c^a, \bar{c}_a) : see Ref.[34] for the details. Then it is easy to obtain the representations in superspace of all the operators listed in (1.3.1)-(1.3.9). We make use of the definition:

$$\delta\Phi^a = [\varepsilon O, \Phi^a] \equiv -\varepsilon \mathcal{O}\Phi^a \quad (1.4.2)$$

where \mathcal{O} is the representation on superspace of the operator O acting on $\widetilde{\mathcal{M}}$ (the commutator in (1.4.2) is the usual commutator in $\widetilde{\mathcal{M}}$). According to (1.4.2) it is not difficult to realize that the representations we are looking for are the following:

$$\mathcal{Q}_{BRS} = -\frac{\partial}{\partial\theta}; \quad \bar{\mathcal{Q}}_{BRS} = \frac{\partial}{\partial\bar{\theta}}; \quad (1.4.3)$$

$$\mathcal{K} = \bar{\theta} \frac{\partial}{\partial\theta}; \quad \bar{\mathcal{K}} = \theta \frac{\partial}{\partial\bar{\theta}}; \quad (1.4.4)$$

$$\mathcal{Q}_g = \bar{\theta} \frac{\partial}{\partial\theta} - \theta \frac{\partial}{\partial\bar{\theta}}; \quad (1.4.5)$$

$$\mathcal{N}_H = \bar{\theta} \frac{\partial}{\partial t}; \quad \bar{\mathcal{N}}_H = \theta \frac{\partial}{\partial t}; \quad (1.4.6)$$

$$\mathcal{Q}_H = -\frac{\partial}{\partial\theta} - \bar{\theta} \frac{\partial}{\partial t}; \quad \bar{\mathcal{Q}}_H = \frac{\partial}{\partial\bar{\theta}} + \theta \frac{\partial}{\partial t}; \quad (1.4.7)$$

$$\tilde{\mathcal{H}} = i \frac{\partial}{\partial t}. \quad (1.4.8)$$

It is easy to check that the algebra among all the operators above is the same as before, the only difference being that here the commutators are given by ordinary derivation with respect to the variables of superspace.

Following the analogy with the relativistic formulation of supersymmetry, we can also construct the covariant derivatives associated to the Susy charges \mathcal{Q}_H and $\bar{\mathcal{Q}}_H$:

$$\mathcal{D}_H = -i \frac{\partial}{\partial\theta} + i\bar{\theta} \frac{\partial}{\partial t} \quad (1.4.9)$$

$$\bar{\mathcal{D}}_H = i \frac{\partial}{\partial\bar{\theta}} - i\theta \frac{\partial}{\partial t}. \quad (1.4.10)$$

which correspond (in $\widetilde{\mathcal{M}}$) to the following operators:

$$D_H = i\mathcal{Q}_{BRS} + i\beta\mathcal{N}_H \quad (1.4.11)$$

$$\bar{D}_H = i\bar{\mathcal{Q}}_{BRS} - i\beta\bar{\mathcal{N}}_H. \quad (1.4.12)$$

Both \mathcal{D}_H , $\overline{\mathcal{D}}_H$ commute with \mathcal{Q}_H , $\overline{\mathcal{Q}}_H$ and \mathcal{H} and they allow to constrain the general superfield (1.4.1) in a Susy-invariant manner. In fact the superfield (1.4.1) is not an irreducible representation of the Susy algebra. On the other hand, if we constrain it imposing that it be annihilated by either $\overline{\mathcal{D}}_H$ or \mathcal{D}_H , we obtain a *chiral* or —respectively— *antichiral* superfield:

$$\Phi_{ch}^a(t, \theta, \overline{\theta}) = \varphi^a + \theta c^a + \overline{\theta} \theta \dot{\varphi}^a; \quad (1.4.13)$$

$$\Phi_{ach}^a(t, \theta, \overline{\theta}) = \varphi^a + \overline{\theta} \omega^{ab} \overline{c}_b - \overline{\theta} \theta \dot{\varphi}^a; \quad (1.4.14)$$

which constitute two inequivalent irreducible representations of the Susy algebra.

The last topic of this section concerns the rewriting of the Hamiltonian $\tilde{\mathcal{H}}$ and the Lagrangian $\tilde{\mathcal{L}}$ in terms of the superfields $\Phi^a(t, \theta, \overline{\theta})$. In fact every supersymmetric theory admits a representation of its action in terms of an integral over the superspace variables of a function of some superfields. This is the case also for the CPI theory. The first thing we notice is that if we rewrite the classical Hamiltonian $H(\varphi)$ in terms of the superfields $\Phi^a(t, \theta, \overline{\theta})$, we obtain:

$$H[\Phi(t, \theta, \overline{\theta})] = H(\varphi) + \theta N_H - \overline{\theta} \overline{N}_H + i\theta \overline{\theta} \tilde{\mathcal{H}}. \quad (1.4.15)$$

This in turn implies that:

$$\tilde{\mathcal{H}} = i \int d\theta d\overline{\theta} H[\Phi(t, \theta, \overline{\theta})]. \quad (1.4.16)$$

The same thing (modulo boundary terms⁷) happens for the Lagrangian (understood as $\frac{1}{2}\varphi^a \omega_{ab} \dot{\varphi}^b - H(\varphi)$):

$$L[\Phi(t, \theta, \overline{\theta})] = L(\varphi) + \theta(\dots) - \overline{\theta}(\dots) + i\theta \overline{\theta} \tilde{\mathcal{L}}; \quad (1.4.17)$$

which implies:

$$\tilde{\mathcal{L}} = i \int d\theta d\overline{\theta} L[\Phi(t, \theta, \overline{\theta})] + (\text{boundary terms}). \quad (1.4.18)$$

This is a nice remark because we can see from Eq.(1.4.17) that both $L(\varphi)$ and $\tilde{\mathcal{L}}(\varphi, c, \lambda, \overline{c})$ belong to the same superfield, i.e. to the same multiplet. But on the other hand we can think of $\int dt L(\varphi)$ as the *quantum weight* (because it is the weight of the trajectories in the Feynman path integral) while $\int dt \tilde{\mathcal{L}}(\varphi, c, \lambda, \overline{c})$ is the *classical weight* (because it plays the same role in the CPI). Therefore, in our context, we can conclude that both Quantum Mechanics and Classical Mechanics belong to the same supermultiplet. How can we pass from one domain to the other? The answer to this question is strictly connected to a limit of the kind $\theta \rightarrow 0$ and $\overline{\theta} \rightarrow 0$, but we will come back on this topic in future works.

⁷The problem of how to properly handle these boundary terms was analyzed in Ref.[2].

2. The Classical Susy

In the previous chapter we have introduced the Classical Path Integral (CPI) and the universal symmetries exhibited by its Lagrangian. We have described the geometrical meaning of some of the symmetry charges, such as the Q_{BRS} and \overline{Q}_{BRS} . In this chapter we want to focus on the meaning of the universal classical Supersymmetry. An attempt in this direction was already made in the past [35], where the authors introduced a nice interplay between the classical Supersymmetry and the concept of ergodicity of a classical system. Here our goal is twofold. First we try to shed some light on the geometrical meaning of the classical Susy (which was not explored in Ref. [35]), and second we develop a little more the ideas put forward in Ref. [35].

To tackle the first issue, i.e. the geometrical aspects of our Susy, the direction we shall take is to make the Susy local and study in detail what we obtain. We say “study in detail” because in the literature there are some strange statements [3] claiming to show that, at least for the supersymmetric QM of Witten [59], the Lagrangian with local-Susy is equivalent to the one with global Susy. We shall show that it is not so. One should actually perform very carefully the full Dirac [22] procedure or, via path-integrals, the Faddeev procedure [24] or apply the BFV methodology [37] of handling systems with constraints. If one does that carefully, it is easy to realize that the system with local Susy has a different number of degrees of freedom than the one with global Susy. The states themselves are restricted to the so called physical states by the presence of the local symmetry. It was this last step that was missing in Ref.[3] and which led to the wrong conclusion.

The physical states condition and the BFV procedure are what will lead us to understand the geometrical meaning of the Susy charges. They will turn out to be an essential ingredient to restrict the forms to the so called equivariant ones [54] [23] [8]. The business of equivariant cohomology has popped out recently in the literature in connection with topological field theories [61]. Some attempts [47] had been done in the past of cooking up a BRS-BFV charge which would produce as physical states the equivariant ones but without showing from which local symmetry this BRS-BFV charge was coming from. Here we shall fill that

gap, that means we shall show in detail which local symmetry gives rise to a BFV charge whose physical states are the equivariant ones.

Next, we shall turn to the other aspect of our supersymmetry, that is its interplay with the concept of ergodicity [35]. To get a better grasp of this problem, it was realized long ago [35] that we had to formulate our functional approach on constant energy surfaces. Therefore we shall modify the CPI-Lagrangian constraining by hand the system to move on some fixed hypersurfaces, the constant energy ones, and check what happens. We will realize that in this formulation the energy plays the role of a coupling and it turns out to be associated to a tadpole term of the new Lagrangian. Moreover we will find that by constraining the system on constant energy surfaces we gain a *local* graded symmetry which is not anyhow a local Susy. Regarding the *global* symmetries we lose part of the original global N=2 Susy which is now reduced to an N=1. This, which may appear as a bad feature of the procedure, may actually turn out to be a virtue. In fact we shall have to study the interplay of ergodicity and Susy by means of only one Susy charge and not two as before. Anyhow the detailed study of this interplay will be left to future works, where we will concentrate more on dynamical issues and not on geometrical ones as we do here.

2.1 Gauging the Global Susy Invariance.

The direction we take to study the geometrical structures behind the supersymmetric charges above is to build a Lagrangian where these symmetries are local. The standard procedure we use is known in the literature [10] as Noether method. It basically consists in finding the exact form of the extra terms generated by *local* variations of the original Lagrangian which had only the global invariances. These extra terms, by Noether theorem, are basically the derivatives of the infinitesimal parameters multiplied by the generators. The trick then is to add to the original Lagrangian a piece made of an *auxiliary field* multiplied by the generator. We can then impose that this auxiliary field transform in such a manner as to cancel the extra variations of the Lagrangian mentioned above.

As the Susy charges are built out of the $Q_{BRS}, \bar{Q}_{BRS}, N_H, \bar{N}_H$ let us build the *local* variations generated by each of these charges on the variables $(\varphi^a, c^a, \lambda_a, \bar{c}_a)$. If we indicate with X one of those four operators and with $(...)$ any of the variables $(\varphi^a, c^a, \lambda_a, \bar{c}_a)$, then by a local variation $\delta_X^{loc}(...)$ we indicate the operation: $\delta_X^{loc}(...) \equiv [\varepsilon(t)X, (...)]$ where now the Grassmannian parameter $\varepsilon(t)$ is dependent on t . These four variations are indicated below:

$$\delta_Q^{loc} \equiv \begin{cases} \delta\varphi^a = \epsilon(t)c^a \\ \delta c^a = 0 \\ \delta\bar{c}_a = i\epsilon(t)\lambda_a \\ \delta\lambda_a = 0 \end{cases} \quad \delta_{\bar{Q}}^{loc} \equiv \begin{cases} \delta\varphi^a = -\bar{\epsilon}(t)\omega^{ab}\bar{c}_b \\ \delta c^a = i\bar{\epsilon}(t)\omega^{ab}\lambda_b \\ \delta\bar{c}_a = 0 \\ \delta\lambda_a = 0 \end{cases} \quad (2.1.1)$$

$$\delta_N^{loc} \equiv \begin{cases} \delta\varphi^a = 0 \\ \delta c^a = 0 \\ \delta\bar{c}_a = \epsilon(t)\partial_a H \\ \delta\lambda_a = i\epsilon(t)c^b\partial_b\partial_a H \end{cases} \quad \delta_{\bar{N}}^{loc} \equiv \begin{cases} \delta\varphi^a = 0 \\ \delta c^a = \bar{\epsilon}(t)\omega^{ab}\partial_b H \\ \delta\bar{c}_a = 0 \\ \delta\lambda_a = i\bar{\epsilon}(t)\bar{c}_d\omega^{db}\partial_b\partial_a H. \end{cases} \quad (2.1.2)$$

We could have used four different parameters for the four different charges (as we will do later on) but here we limit ourselves just to two: $\epsilon(t)$ and $\bar{\epsilon}(t)$. The local Susy variations associated to the two Susy charges of Eqs.(1.3.8) and (1.3.9) are:

$$\begin{cases} \delta_{Q_H}^{loc} = \delta_Q^{loc} - \beta\delta_N^{loc} \\ \delta_{\bar{Q}_H}^{loc} = \delta_{\bar{Q}}^{loc} + \beta\delta_{\bar{N}}^{loc}. \end{cases} \quad (2.1.3)$$

It is now straightforward to check that the local Susy variations of the Lagrangian $\tilde{\mathcal{L}}$ of Eq.(1.1.6) give the following results:

$$\delta_{Q_H}^{loc} \tilde{\mathcal{L}} = -i\dot{\epsilon}Q_H + (t.d.) \quad (2.1.4)$$

and

$$\delta_{\bar{Q}_H}^{loc} \tilde{\mathcal{L}} = -i\dot{\bar{\epsilon}}\bar{Q}_H + (t.d.). \quad (2.1.5)$$

With $(t.d.)$ we indicate total derivative terms: they turn into surface terms in the action and they disappear if we require that $\epsilon(t)$ and $\bar{\epsilon}(t)$ be zero at the end points of integrations as we will do from now on. To do things in a cleaner manner we should have actually checked the invariance using the integrated charge as explained in Appendix A.1. Anyhow from Eqs.(2.1.4) and (2.1.5) we see that the Lagrangian does not change by a total derivative so the two local Susy transformations are not symmetries and we have to modify the Lagrangian to find another one which is invariant. If we find it, then it must also be invariant [10] under the composition of two local Susy transformations which we can prove (see Appendix A.2) to be the sum of a local supersymmetry transformation plus a *local* time-translation generated by $\tilde{\mathcal{H}}$. This last one is not a symmetry of $\tilde{\mathcal{L}}$ and the Lagrangian would change by a term proportional to $\tilde{\mathcal{H}}$ multiplied by the time-derivative of the symmetry parameter, exactly as the Noether theorem requires. The trick [10] to get the invariance is to add to $\tilde{\mathcal{L}}$ some auxiliary fields multiplied by the charges under which $\tilde{\mathcal{L}}$ is not invariant. In our case the complete Lagrangian is:

$$\tilde{\mathcal{L}}_{Susy} \equiv \tilde{\mathcal{L}} + \bar{\psi}Q_H + \psi\bar{Q}_H + g\tilde{\mathcal{H}}, \quad (2.1.6)$$

where $g(t), \psi(t), \bar{\psi}(t)$ are three new fields (the last two of Grassmannian nature) whose variations under the local Susy will be determined by the requirement that $\tilde{\mathcal{L}}_{Susy}$ be invariant under the local Susy variations of Eq.(2.1.3). In detail we get:

$$\delta_{Q_H} \tilde{\mathcal{L}}_{Susy} = -i\dot{\epsilon} Q_H + (\delta_{Q_H} g) \tilde{\mathcal{H}} + (\delta_{Q_H} \bar{\psi}) Q_H + (\delta_{Q_H} \psi) \bar{Q}_H + \psi(2i\epsilon\beta\tilde{\mathcal{H}}) \quad (2.1.7)$$

and we see that the following transformations for the variables $g, \psi, \bar{\psi}$ would make $\tilde{\mathcal{L}}_{Susy}$ invariant under the local transformation associated to Q_H :

$$\begin{cases} \delta_{Q_H} \bar{\psi} = i\dot{\epsilon} \\ \delta_{Q_H} \psi = 0 \\ \delta_{Q_H} g = +2i\epsilon\beta\psi. \end{cases} \quad (2.1.8)$$

For the variation under \bar{Q}_H we get:

$$\delta_{\bar{Q}_H} \tilde{\mathcal{L}}_{Susy} = -i\dot{\bar{\epsilon}} \bar{Q}_H + (\delta_{\bar{Q}_H} g) \tilde{\mathcal{H}} + (\delta_{\bar{Q}_H} \bar{\psi}) Q_H + (\delta_{\bar{Q}_H} \psi) \bar{Q}_H + \bar{\psi}(2i\bar{\epsilon}\beta\tilde{\mathcal{H}}) \quad (2.1.9)$$

and we see that the following transformations for the variables $g, \psi, \bar{\psi}$ make $\tilde{\mathcal{L}}_{Susy}$ invariant under the local transformation associated to \bar{Q}_H :

$$\begin{cases} \delta_{\bar{Q}_H} \bar{\psi} = 0 \\ \delta_{\bar{Q}_H} \psi = i\dot{\bar{\epsilon}} \\ \delta_{\bar{Q}_H} g = +2i\bar{\epsilon}\beta\bar{\psi}. \end{cases} \quad (2.1.10)$$

Last we should check how $\tilde{\mathcal{L}}$ changes under a local time-reparametrization. We have to do that because this reparametrization appears in the composition of two local Susy transformations (Appendix A.2). The action of the local time reparametrization on the variables $(\varphi^a, c^a, \lambda_a, \bar{c}_a)$ is listed in formula (A.2.9) of Appendix A.2. Under those transformations we can easily prove that

$$\delta \tilde{\mathcal{L}} = -i\dot{\eta} \tilde{\mathcal{H}} \quad (2.1.11)$$

where $\eta(t)$ is the time-dependent parameter of the transformation. Let us now use this result in the variation of $\tilde{\mathcal{L}}_{Susy}$ under time reparametrization:

$$\delta \tilde{\mathcal{L}}_{Susy} = -i\dot{\eta} \tilde{\mathcal{H}} + \delta \bar{\psi} Q_H + \delta \psi \bar{Q}_H + \delta g \tilde{\mathcal{H}}. \quad (2.1.12)$$

We immediately notice that $\tilde{\mathcal{L}}_{Susy}$ is invariant under the local time-reparametrization if we transform the variables $(\psi, \bar{\psi}, g)$ as follows

$$\begin{cases} \delta \psi = \delta \bar{\psi} = 0 \\ \delta g = i\dot{\eta}. \end{cases} \quad (2.1.13)$$

So, putting together all the three local symmetries, (2.1.8), (2.1.10) and (2.1.13), we can say that $\tilde{\mathcal{L}}_{Susy}$ (if we choose $\beta = 1$ in Eqs.(2.1.3)-(2.1.10)) has the following local invariance:

$$\begin{cases} \delta\psi = i\dot{\bar{\epsilon}} \\ \delta\bar{\psi} = i\dot{\epsilon} \\ \delta g = i\dot{\eta} + 2i(\bar{\epsilon}\bar{\psi} + \epsilon\psi). \end{cases} \quad (2.1.14)$$

It is not the first time that one-dimensional systems with local-Susy have been built. The first work was the classic one of Brink et al. [10]. Later on people [3] played with the supersymmetric Quantum Mechanical model (Susy-QM) of Witten [59] turning its global Susy into a local invariance. Regarding this model, the author of Ref.[3] pretended to show that the locally Supersymmetric Quantum Mechanics was equivalent to the standard Susy QM with only global invariance. The proof was based on the fact that, via the analog of the transformations (2.1.14), it is possible to bring the variables $(\psi, \bar{\psi}, g)$ to zero and so, looking at Eq.(2.1.6), this would imply that we can turn $\tilde{\mathcal{L}}_{Susy}$ into $\tilde{\mathcal{L}}$. This kind of reasoning is *misleading*. In fact, while it is easy to check (see Appendix A.4) that it is possible to bring the $(\psi, \bar{\psi}, g)$ to zero via the transformations (2.1.14), it should be remembered that the starting point was a gauge theory, $\tilde{\mathcal{L}}_{Susy}$, and the value zero for the variables $(\psi, \bar{\psi}, g)$ is equivalent to a particular choice of gauge-fixing. Anyhow, the *physical* theory has to be gauge-fixing independent and this is achieved [37] by restricting the physical states via the BRS charge associated to the local symmetries. So in the gauge-fixing where the $(\psi, \bar{\psi}, g)$ are zero the locally-supersymmetric QM [3] has the same action as the globally supersymmetric theory [59] but we have to restrict the states to the physical ones which are basically those annihilated by the symmetry charges. So the two systems, the one with global Susy and the one with local Susy, are not equivalent even if they seem to be so in a particular gauge-fixing. Their Hilbert spaces are different even if the dynamics, in a particular gauge-fixing, is the same. Moreover, even concerning the degrees of freedom, we shall show that while $\tilde{\mathcal{L}}$ has $8n$ independent variables the $\tilde{\mathcal{L}}_{Susy}$ has $8n - 6$. To do this analysis we shall go through the business of studying the constraints associated to the local symmetries of $\tilde{\mathcal{L}}_{Susy}$.

The standard procedure is the one of Dirac [22] which we will follow here in detail. Looking at $\tilde{\mathcal{L}}_{Susy}$ we see that the primary constraints are:

$$\begin{cases} \Pi_\psi = 0 \\ \Pi_{\bar{\psi}} = 0 \\ \Pi_g = 0 \end{cases} \quad (2.1.15)$$

where Π_ψ , $\Pi_{\bar{\psi}}$ and Π_g are the momenta associated⁸ to ψ , $\bar{\psi}$ and g . The *canonical* Hamiltonian [22] is then the following:

⁸Remember that here and in the sequel we shall use right derivatives to define the momenta

$$\tilde{\mathcal{H}}_{can.} = \tilde{\mathcal{H}}_{Susy} = \tilde{\mathcal{H}} - \psi \overline{Q}_H - \overline{\psi} Q_H - g \tilde{\mathcal{H}} \quad (2.1.16)$$

while the *primary* [22] (or total) Hamiltonian is:

$$\tilde{\mathcal{H}}_P = (1 - g) \tilde{\mathcal{H}} - \psi \overline{Q}_H - \overline{\psi} Q_H + u_1 \Pi_\psi + u_2 \Pi_{\overline{\psi}} + u_3 \Pi_g \quad (2.1.17)$$

and it is obtained by adding to $\tilde{\mathcal{H}}_{can.}$ the primary constraints (2.1.15) via the Lagrange multipliers u_1, u_2, u_3 . Next we have to impose that the primary constraints do not change under the time evolution, i.e:

$$[\Pi_\psi, \tilde{\mathcal{H}}_P] = 0, \quad [\Pi_{\overline{\psi}}, \tilde{\mathcal{H}}_P] = 0, \quad [\Pi_g, \tilde{\mathcal{H}}_P] = 0. \quad (2.1.18)$$

Here we have used the commutators instead of the Extended-Poisson-Brackets. We did that because the two structures are isomorphic as explained in Section 1.2. In particular the (graded) commutators we need in (2.1.18) are

$$[\Pi_\psi, \psi] = 1, \quad [\Pi_{\overline{\psi}}, \overline{\psi}] = 1, \quad [\Pi_g, g] = -i. \quad (2.1.19)$$

Using them we get from (2.1.18) the following set of *secondary* [22] constraints:

$$\begin{cases} \overline{Q}_H = 0 \\ Q_H = 0 \\ \tilde{\mathcal{H}} = 0. \end{cases} \quad (2.1.20)$$

At this point the careful reader could ask which are the operators generating the full set of transformations (2.1.14), especially the last one. It does not seem that they are generated by the operators of Eqs. (2.1.20) and (2.1.15). Actually the answer to this question is rather subtle and tricky [37] and is given in full details in Appendix A.3.

Having clarified this point, we can go on with our procedure. We have now to require that also the secondary constraints (2.1.20) do not change under time evolution using as operator of evolution always the primary Hamiltonian $\tilde{\mathcal{H}}_P$ as explained in Ref. [22]. It is easy to realize that in our case we do not generate further constraints with this procedure and that, at the same time, we do not determine the Lagrange multipliers. The fact that the Lagrange multipliers are all left undetermined is a signal that the constraints are first class [22] as it is easy to check by doing the commutators among all the six constraints (2.1.20) and (2.1.15). Being them first class, one has to introduce six gauge-fixings which will be used to determine the Lagrange multipliers [22].

of the Grassmannian variables: $\Pi_\psi = \frac{\overleftarrow{\partial} \tilde{\mathcal{L}}}{\partial \psi}$.

The gauge-fixings — let us call them χ_i — must have a non-zero commutator with the associated gauge generator. For the three constraints of Eq.(2.1.15) three suitable gauge-fixings can be:

$$\psi - \psi_0 = 0, \quad \bar{\psi} - \bar{\psi}_0 = 0, \quad g - g_0 = 0, \quad (2.1.21)$$

where ψ_0 , $\bar{\psi}_0$ and g_0 are three fixed functions. It is easy to check that each of them does not commute with its associated generator. The gauge-fixing $\psi_0 = \bar{\psi}_0 = g_0 = 0$ is among the admissible ones, in the sense that there is a gauge transformation which brings any configuration into this one as shown in Appendix A.4. In this gauge fixing we get that the $(\psi, \bar{\psi}, g)$ variables disappear from the $\tilde{\mathcal{L}}_{Susy}$ and so $\tilde{\mathcal{L}}_{Susy}$ apparently is reduced to $\tilde{\mathcal{L}}$. Of course, as we said earlier, this should not mislead us to think that the physics of $\tilde{\mathcal{L}}_{Susy}$ is the same as the one of $\tilde{\mathcal{L}}$. In fact at the Hamiltonian level, even if $\tilde{\mathcal{H}}_{Susy}$ is reduced to $\tilde{\mathcal{H}}$ by the gauge-fixing, the Poisson Brackets for the two systems are different. For the one with local symmetries the Poisson Brackets are the Dirac ones which, given two observables O_1 and O_2 , are built as:

$$\{O_1, O_2\}_{DB} = \{O_1, O_2\} - \{O_1, G_i\}(C^{-1})^{ij}\{G_j, O_2\}, \quad (2.1.22)$$

We have indicated with G_i any of the six first class constraints of Eqs.(2.1.20) and (2.1.15), and the matrix C_{ij} has its elements built as $\{G_i, \chi_j\}$ where χ_i are the six gauge-fixings associated to the constraints G_i . The brackets entering the expressions on the RHS of (2.1.22) are the standard Extended Poisson Brackets of Eq.(1.2.1). If the dynamics is the one of a system with global Susy only, that is one whose Hamiltonian is really $\tilde{\mathcal{H}}$ from the beginning, then the Poisson Brackets among the same two observables O_1 and O_2 would be $\{O_1, O_2\}$ and it is clear that

$$\{O_1, O_2\} \neq \{O_1, O_2\}_{DB}. \quad (2.1.23)$$

This explains why, even if the Hamiltonians of the two systems (the one with local symmetries and the one with global ones) are the same (in some gauge-fixings), the two dynamics are anyhow different because they are ruled by different Poisson Brackets.

Also the counting of the degrees of freedom indicates that the systems have different numbers of degrees of freedom. The one with global symmetries, and Lagrangian $\tilde{\mathcal{L}}$, has just the variables $(\varphi^a, c^a, \lambda_a, \bar{c}_a)$ which are $8n$. The one with local symmetries, $\tilde{\mathcal{L}}_{Susy}$, has the variables $(\varphi^a, c^a, \lambda_a, \bar{c}_a)$ which are $8n$, plus the three gauge variables $(\psi, \bar{\psi}, g)$ and the relative momenta for a total of 6, minus the 6 constraints of Eqs.(2.1.15)(2.1.20), minus the 6 gauge fixings χ_i , for a total of $8n - 6$ variables. This is the correct counting of variables as explained in Ref. [37]. So even from this we realize that the two systems are different. As we said at the

beginning, the classical path-integral is actually the counterpart of the operatorial version of CM proposed by Koopman and von Neumann [42] and if we adopt this operatorial version we should use the commutators derived in Eq.(1.2.3). This operatorial version of course could be adopted also for the dynamics with local symmetries associated to $\tilde{\mathcal{H}}_{Susy}$. In the operatorial formulation we have a Hilbert space but we know that, for a system with local symmetries, the Hilbert space is restricted to the *physical states*. The selection of these states is done by a BRS-BFV charge [37] associated to the gauge-symmetries of the system. This BRS-BFV charge of course has nothing to do with the Q_{BRS} of Eq.(1.3.1). The construction of the BRS-BFV charge for our $\tilde{\mathcal{L}}_{Susy}$ goes as follows [37]. First we should introduce a pair of *gauge* ghost-antighosts for each gauge generator. As our gauge generators are

$$G_i = (\Pi_{\bar{\psi}}, \Pi_{\psi}, \Pi_g, Q_H, \bar{Q}_H, \tilde{\mathcal{H}}) \quad (2.1.24)$$

the ghost-antighosts are twelve and can be indicated as:

$$\begin{aligned} \eta^i &= (\eta_{\bar{\psi}}, \eta_{\psi}, \eta_g, \eta_H, \bar{\eta}_H, \tilde{\eta}_H) \\ \mathcal{P}_i &= (\mathcal{P}_{\bar{\psi}}, \mathcal{P}_{\psi}, \mathcal{P}_g, \mathcal{P}_H, \bar{\mathcal{P}}_H, \tilde{\mathcal{P}}_H). \end{aligned} \quad (2.1.25)$$

The general BRS-BFV charge [37] is then⁹:

$$\Omega_{BRS} = \eta^i G_i - \frac{1}{2} (-)^{\epsilon_i} \eta^i \eta^j C_{ji}^k \mathcal{P}_k, \quad (2.1.26)$$

where the ϵ_i is the Grassmannian grading of the constraints G_i and C_{ij}^k are the structure constants of the algebra of our constraints. In our case this algebra is:

$$\begin{aligned} [Q_H, \bar{Q}_H] &= 2i\tilde{\mathcal{H}} \\ [Q_H, Q_H] &= [\bar{Q}_H, \bar{Q}_H] = [Q_H, \tilde{\mathcal{H}}] = [\bar{Q}_H, \tilde{\mathcal{H}}] = 0 \end{aligned} \quad (2.1.27)$$

where we have put $\beta = 1$ with respect to Eq.(1.3.10).

It is now easy to work out the BRS-BFV charge for our local Susy invariance:

$$\Omega_{BRS}^{(Susy)} = \eta_{\bar{\psi}} \Pi_{\bar{\psi}} + \eta_{\psi} \Pi_{\psi} + \eta_g \Pi_g + \eta_H Q_H + \bar{\eta}_H \bar{Q}_H + \tilde{\eta}_H \tilde{\mathcal{H}} - 2i\eta_H \bar{\eta}_H \tilde{\mathcal{P}}_H \quad (2.1.28)$$

Note that it contains terms with three ghosts and so it is hard to see how it works on the states. These terms with three ghosts are there because the generators are not in involution. As it is explained in Ref. [37] in case the constraints G_i are not in involution, one can build some new ones $F_i = a_i^j G_j$ which are in involution. In our case the F_i generators, replacing the Q_H and \bar{Q}_H , can be easily worked out and they are:

⁹The graded commutators among the ghosts of (2.1.25) are $[\eta^i, \mathcal{P}_j] = \delta_j^i$

$$\begin{cases} Q_A \equiv (Q_H + \psi \tilde{\mathcal{H}}) \\ Q_B \equiv (\bar{Q}_H - 2i\Pi_\psi) \end{cases} \quad (2.1.29)$$

while the other F_i are the same as the G_j . The associated BRS-BFV charge is then

$$\Omega_{BRS}^{(F)} = \eta_{\bar{\psi}} \Pi_{\bar{\psi}} + \eta_\psi \Pi_\psi + \eta_g \Pi_g + \eta_A Q_A + \eta_B Q_B + \tilde{\eta}_H \tilde{\mathcal{H}}. \quad (2.1.30)$$

We have called η_A, η_B the BFV ghosts associated to Q_A, Q_B . Note that this $\Omega_{BRS}^{(F)}$ does not contain terms with three ghosts. The *physical states* are then defined as

$$\Omega_{BRS}^{(F)} | \text{phys} \rangle = 0 \quad (2.1.31)$$

and, by following Ref. [37], we can easily show that (2.1.31) is equivalent to the following six constraints:

$$(1) \begin{cases} \Pi_{\bar{\psi}} | \text{phys} \rangle = 0 \\ \Pi_\psi | \text{phys} \rangle = 0 \\ \Pi_g | \text{phys} \rangle = 0 \end{cases} \quad (2) \begin{cases} Q_A | \text{phys} \rangle = 0 \\ Q_B | \text{phys} \rangle = 0 \\ \tilde{\mathcal{H}} | \text{phys} \rangle = 0. \end{cases} \quad (2.1.32)$$

The set (1) above means that the physical states must be independent of the $(\psi, \bar{\psi}, g)$ that means independent of any choice of gauge fixing. The set (2) instead (combined with some of the conditions from the set (1)) is equivalent to the following conditions:

$$\begin{aligned} Q_H | \text{phys} \rangle &= 0 \\ \bar{Q}_H | \text{phys} \rangle &= 0 \\ \tilde{\mathcal{H}} | \text{phys} \rangle &= 0. \end{aligned} \quad (2.1.33)$$

We can summarize it by saying that, even if in some gauge-fixing the $\tilde{\mathcal{H}}_{Susy}$ is the same as $\tilde{\mathcal{H}}$, the dynamics of the first is restricted to a subset (given by Eq.(2.1.33)) of the full-Hilbert space while the dynamics of $\tilde{\mathcal{H}}$ is not restricted. This is what is not spelled out correctly in Ref. [3]. The author may have been brought to the wrong conclusions not only because he did not consider the correct Hilbert space but also by the following fact that we like to draw to the attention of the reader. The three quantities $Q_H, \bar{Q}_H, \tilde{\mathcal{H}}$ entering the constraints (2.1.20) are actually constants of motion in the space $(\varphi^a, c^a, \lambda_a, \bar{c}_a)$. So fixing them, as the constraints do, is basically fixing a set of initial conditions. The hypersurface defined by (2.1.20) is the subspace of $(\varphi^a, c^a, \lambda_a, \bar{c}_a)$ where the motion takes place and it takes place with the same dynamics as in $(\varphi^a, c^a, \lambda_a, \bar{c}_a)$. If the constraint surfaces were not made by constants of motion, then the dynamics would have to be modified to force the particle to move on them, but this is not the case here.

2.2 New Local Susy Invariance and Equivariance

In Chapter 1 we have seen that, besides the Susy, there are other global invariances of \mathcal{H} . In the following lines we show what happens when we gauge separately the Q_{BRS} , \bar{Q}_{BRS} , N_H , \bar{N}_H of Eqs.(1.3.1),(1.3.2), (1.3.6) and (1.3.7). The local variation of $\tilde{\mathcal{L}}$ under these four combined gauge transformations is

$$\begin{aligned}\delta^{loc.}\tilde{\mathcal{L}} &= [\epsilon Q_{BRS} + \bar{\epsilon}\bar{Q}_{BRS} + \eta N_H + \bar{\eta}\bar{N}_H, \tilde{\mathcal{L}}] \\ &= -i\dot{\epsilon}Q_{BRS} - i\dot{\bar{\epsilon}}\bar{Q}_{BRS} - i\dot{\eta}N_H - i\dot{\bar{\eta}}\bar{N}_H,\end{aligned}\quad (2.2.1)$$

where $(\epsilon(t), \bar{\epsilon}(t), \eta(t), \bar{\eta}(t))$ are four different Grassmannian gauge parameters. From equation (2.2.1) one could be tempted to propose the following as *extended* Lagrangian invariant under the local symmetries above:

$$\tilde{\mathcal{L}}_{ext.} \equiv \tilde{\mathcal{L}} + \alpha(t)Q_{BRS} + \bar{\alpha}(t)\bar{Q}_{BRS} + \beta(t)N_H + \bar{\beta}(t)\bar{N}_H, \quad (2.2.2)$$

where $(\alpha(t), \bar{\alpha}(t), \beta(t), \bar{\beta}(t))$ are four Grassmannian gauge-fields which we could transform in a proper way in order to make $\tilde{\mathcal{L}}_{ext.}$ invariant. This is actually impossible whatever transformation we envision for the gauge-fields. In fact, as we did in the case of the Susy of the previous section, we should consider what happens when we compose two of the local symmetries of Eq.(2.2.1). This information is given by the following commutators [34]:

$$[Q_{BRS}, \bar{N}_H] = i\tilde{\mathcal{H}}; \quad [\bar{Q}_{BRS}, N_H] = -i\tilde{\mathcal{H}}. \quad (2.2.3)$$

This basically tells us that we should add to the Lagrangian $\tilde{\mathcal{L}}_{ext.}$ of Eq.(2.2.2) an extra gauge field $g(t)$ and an extra gauge generator¹⁰ $\tilde{\mathcal{H}}$:

$$\tilde{\mathcal{L}}_{ext.} = \tilde{\mathcal{L}} + \alpha(t)Q_{BRS} + \bar{\alpha}(t)\bar{Q}_{BRS} + \beta(t)N_H + \bar{\beta}(t)\bar{N}_H + g(t)\tilde{\mathcal{H}}. \quad (2.2.4)$$

Doing now an extended gauge transformation like in (2.2.1) we get:

$$\begin{aligned}\delta_{loc.}\tilde{\mathcal{L}}_{ext.} &= -i\dot{\epsilon}Q_{BRS} - i\dot{\bar{\epsilon}}\bar{Q}_{BRS} - i\dot{\eta}N_H - i\dot{\bar{\eta}}\bar{N}_H + (\delta\alpha)Q_{BRS} \\ &\quad + i\alpha\dot{\eta}\tilde{\mathcal{H}} + (\delta\bar{\alpha})\bar{Q}_{BRS} - i\bar{\alpha}\dot{\eta}\tilde{\mathcal{H}} + (\delta\beta)N_H \\ &\quad - i\beta\dot{\bar{\epsilon}}\tilde{\mathcal{H}} + (\delta\bar{\beta})\bar{N}_H + i\bar{\beta}\dot{\epsilon}\tilde{\mathcal{H}} + (\delta g)\tilde{\mathcal{H}},\end{aligned}\quad (2.2.5)$$

¹⁰We will indicate with greek letters the gauge fields associated to Grassmannian generators and with latin letters the one associated to bosonic generators.

and from this it is easy to see that $\tilde{\mathcal{L}}_{ext.}$ is invariant if the gauge-fields $(\alpha, \bar{\alpha}, \beta, \bar{\beta}, g)$ are transformed as follows:

$$\begin{cases} \delta\alpha = i\dot{\epsilon} \\ \delta\bar{\alpha} = i\dot{\bar{\epsilon}} \\ \delta\beta = i\dot{\eta} \\ \delta\bar{\beta} = i\dot{\bar{\eta}} \\ \delta g = i\bar{\alpha}\dot{\eta} - i\alpha\dot{\bar{\eta}} + i\beta\dot{\bar{\epsilon}} - i\bar{\beta}\dot{\epsilon} . \end{cases} \quad (2.2.6)$$

From these transformations we notice that, for some choice of the gauge-fields and of the gauge-transformations, we do not need to have the $g\tilde{\mathcal{H}}$ in the $\tilde{\mathcal{L}}_{ext.}$ The first choice is $\alpha = \bar{\alpha} = \epsilon = \bar{\epsilon} = 0$ which, from Eq.(2.2.6), implies that we can choose $g(t) = 0$. The $\tilde{\mathcal{L}}_{ext.}$ is then

$$\tilde{\mathcal{L}}_N \equiv \tilde{\mathcal{L}} + \beta(t)N_H + \bar{\beta}(t)\bar{N}_H. \quad (2.2.7)$$

The second choice, which also implies that we can choose $g(t) = 0$, is $\beta = \bar{\beta} = \eta = \bar{\eta} = 0$ and this would lead to the following Lagrangian

$$\tilde{\mathcal{L}}_{Q_{BRS}} \equiv \tilde{\mathcal{L}} + \alpha(t)Q_{BRS} + \bar{\alpha}(t)\bar{Q}_{BRS}. \quad (2.2.8)$$

We shall hang around here for a moment spending some time on the Lagrangian $\tilde{\mathcal{L}}_{Q_{BRS}}$ above because it allows us to do some crucial observations on the counting of the degrees of freedom. As we said in the previous section the Lagrangian $\tilde{\mathcal{L}}_{Susy}$ of Eq.(2.1.6) has fewer degrees of freedom than the standard one $\tilde{\mathcal{L}}$ of Eq.(1.1.6), and the same happens with $\tilde{\mathcal{L}}_{Q_{BRS}}$. In fact in $\tilde{\mathcal{L}}_{Q_{BRS}}$ we have two primary constraints:

$$\Pi_\alpha = 0; \quad \Pi_{\bar{\alpha}} = 0; \quad (2.2.9)$$

which generate two secondary ones:

$$Q_{BRS} = 0; \quad \bar{Q}_{BRS} = 0. \quad (2.2.10)$$

All these four constraints are first class so we need four gauge-fixings. The total counting [37] is then $8n$ (original variables) $+4$ (gauge variables and momenta) -4 (constraints) -4 (gauge-fixings) for a total of $8n - 4$ phase-space variables. At this point one question that arises naturally is: “*Is it possible to have a Lagrangian with the local invariances generated by Q_{BRS} and \bar{Q}_{BRS} , but with the same number of degrees of freedom as $\tilde{\mathcal{L}}$?*”. The answer is yes. In fact let us start from the following Lagrangian:

$$\tilde{\mathcal{L}}'_{Q_{BRS}} \equiv \tilde{\mathcal{L}} + \dot{\alpha}(t)Q_{BRS} + \dot{\bar{\alpha}}(t)\bar{Q}_{BRS}. \quad (2.2.11)$$

It is easy to check that it is invariant under the following set of local transformations generated by the Q_{BRS} and \overline{Q}_{BRS} :

$$\begin{aligned}\delta(\dots) &= [\epsilon(t)Q_{BRS} + \bar{\epsilon}(t)\overline{Q}_{BRS}, (\dots)] \\ \delta\alpha &= i\epsilon \\ \delta\overline{\alpha} &= i\bar{\epsilon};\end{aligned}\tag{2.2.12}$$

where we have indicated with (\dots) any of the variables $(\varphi^a, c^a, \lambda_a, \bar{c}_a)$. Anyhow at the same time it is easy to check that the Lagrangian $\tilde{\mathcal{L}}'_{Q_{BRS}}$ of (2.2.11) has only two primary constraints¹¹

$$\begin{aligned}\Pi_\alpha + Q_{BRS} &= 0; \\ \Pi_{\bar{\alpha}} + \overline{Q}_{BRS} &= 0;\end{aligned}\tag{2.2.13}$$

and no secondary ones. The above two constraints are first class so we need just two gauge-fixings and not four as before. The counting of degrees of freedom now goes as follows: $8n$ (original variables)+4(gauge variables and momenta)–2 (constraints)–2 (gauge fixings) for a total of $8n$ variables. So we see that the system described by the Lagrangian $\tilde{\mathcal{L}}'_{Q_{BRS}}$ of Eq.(2.2.11) has the same number of degrees of freedom as the original $\tilde{\mathcal{L}}$. In Appendix A.5 we will show how the constraints (2.2.13) act in the Hilbert space of the system. The fact that $\tilde{\mathcal{L}}'_{Q_{BRS}}$ and $\tilde{\mathcal{L}}$ are somehow equivalent could also be understood by doing an integration by parts of the terms of Eq.(2.2.11) containing $\dot{\alpha}$ and $\dot{\bar{\alpha}}$. The integration by parts produces, with respect to $\tilde{\mathcal{L}}$, some terms which vanish because of the conservation of Q_{BRS} and \overline{Q}_{BRS} . Moreover we should mention that it is not the first time people thought of making the BRS-antiBRS invariance local [6]. We will come back in Ref. [20] to the issue of gauging the BRS symmetry and all the $\mathfrak{isp}(2)$ charges of Eqs.(1.3.1)-(1.3.9).

Let us now return to Eq.(2.2.4). The two choices which led to Eqs.(2.2.7) and (2.2.8) are not the only ones consistent with the transformations (2.2.6). Another choice is

$$\begin{cases} \alpha = -\bar{\beta} \\ \bar{\alpha} = \beta \end{cases} \quad \begin{cases} \epsilon = -\bar{\eta} \\ \bar{\epsilon} = \eta \end{cases} .\tag{2.2.14}$$

With this choice the Lagrangian that we get from (2.2.4) is:

$$\tilde{\mathcal{L}}_{eq} \equiv \tilde{\mathcal{L}} + \alpha(t)Q_{(1)} + \bar{\alpha}(t)Q_{(2)} + g(t)\tilde{\mathcal{H}}.\tag{2.2.15}$$

The suffix “eq” on the Lagrangian stands for “equivariant” and the reason will be clear in a while. The $Q_{(1)}, Q_{(2)}$ on the RHS of Eq. (2.2.15) are

¹¹Remember that we use right derivatives in the definition of the momenta: $\Pi_\alpha \equiv \frac{\overleftarrow{\partial} \tilde{\mathcal{L}}'_{Q_{BRS}}}{\partial \dot{\alpha}}$.

$$\begin{cases} Q_{(1)} \equiv Q_{BRS} - \overline{N}_H \\ Q_{(2)} \equiv \overline{Q}_{BRS} + N_H. \end{cases} \quad (2.2.16)$$

Note that these charges, with respect to the Q_H and \overline{Q}_H of Eqs.(1.3.8) and (1.3.9), are somehow twisted in the sense that here we sum the Q_{BRS} with \overline{N} and not with N and viceversa for the \overline{Q}_{BRS} . It is easy to check that:

$$Q_{(1)}^2 = Q_{(2)}^2 = -i\tilde{\mathcal{H}}; \quad [Q_{(1)}, Q_{(2)}] = 0. \quad (2.2.17)$$

So these two charges generate two supersymmetry transformations which are anyhow different from those generated by the supersymmetry generators of Eqs.(1.3.8) and (1.3.9). The Lagrangian of Eq.(2.2.15) has two *local* Susy invariances but different from the ones of $\tilde{\mathcal{L}}_{Susy}$ (2.1.6). In order to get the Lagrangian (2.1.6) we should have made in Eq.(2.2.4) and (2.2.6) the following choice:

$$\begin{cases} \alpha = -\beta = \overline{\psi} \\ \overline{\alpha} = \overline{\beta} = \psi \end{cases} \quad \begin{cases} \epsilon = -\eta \\ \overline{\epsilon} = \overline{\eta}. \end{cases} \quad (2.2.18)$$

Going back to Eq.(2.2.15), let us now restrict the Lagrangian to the following one:

$$\tilde{\mathcal{L}}_{eq} = \tilde{\mathcal{L}} + \alpha(t)Q_{(1)} + g(t)\tilde{\mathcal{H}}, \quad (2.2.19)$$

which is locally invariant only under one Susy and the symmetry transformations are:

$$\begin{cases} \delta(\dots) = [\epsilon Q_{(1)} + \tau \tilde{\mathcal{H}}, (\dots)] \\ \delta\alpha = i\dot{\epsilon} \\ \delta g = 2i\alpha\epsilon + i\dot{\tau}, \end{cases} \quad (2.2.20)$$

where (...) indicates any of the variables $(\varphi^a, c^a, \lambda_a, \overline{c}_a)$ and $\epsilon(t)$ and $\tau(t)$ are infinitesimal parameters. Because of this gauge invariance, we have to handle the system either via the Faddeev procedure [24] or the BFV method [37]. We will follow this last one. The constraints (primary and secondary) derived from (2.2.19) are:

$$\begin{cases} \Pi_\alpha = 0 \\ \Pi_g = 0 \\ Q_{(1)} = 0 \\ \tilde{\mathcal{H}} = 0 \end{cases} \quad (2.2.21)$$

where Π_α and Π_g are respectively the momenta conjugate to the gauge fields $\alpha(t)$ and $g(t)$. The BFV procedure, as explained in the previous section, tells us to

add four new ghosts and their respective momenta to the system. We will indicate them as follows:

$$\left\{ \begin{array}{l} (C^{(1)}, C^H, \overline{C}_{(1)}, \overline{C}_H) \\ (\overline{\mathcal{P}}_{(1)}, \overline{\mathcal{P}}_H, \mathcal{P}_{(1)}, \mathcal{P}_H) \end{array} \right. \quad (2.2.22)$$

We shall impose the following graded-commutators:

$$\left\{ \begin{array}{l} [g, \Pi_g] = [C^{(1)}, \overline{\mathcal{P}}_{(1)}] = [\overline{C}_{(1)}, \mathcal{P}_{(1)}] = 1 \\ [\alpha, \Pi_\alpha] = [C^H, \overline{\mathcal{P}}_H] = [\overline{C}_H, \mathcal{P}_H] = 1 \end{array} \right. \quad (2.2.23)$$

In the first line above the variables are all “bosonic” while in the second one are all Grassmannian. Equipped with all these tools we will now build the BFM-BRS [37] charge associated to our constraints:

$$\Omega_{BRS}^{(Eq.)} \equiv C^{(1)}Q_{(1)} + C^H\tilde{\mathcal{H}} + \mathcal{P}_{(1)}\Pi_\alpha + \mathcal{P}_H\Pi_g + i(C^{(1)})^2\overline{\mathcal{P}}_H \quad (2.2.24)$$

It is easy to check that $(\Omega_{BRS}^{(Eq.)})^2 = 0$ as a BRS charge should be. The next step, analogous to what we did in Section 2.1, is to select as physical states those annihilated by the $\Omega_{BRS}^{(eq.)}$ charge:

$$\Omega_{BRS}^{(Eq.)} | \text{phys} \rangle = 0 \quad (2.2.25)$$

Because of the nilpotent character of the $\Omega_{BRS}^{(Eq.)}$, we should remember that two physical states are equivalent if they differ by a BRS variation:

$$| \text{phys-2} \rangle = | \text{phys-1} \rangle + \Omega_{BRS}^{(Eq.)} | \chi \rangle \quad (2.2.26)$$

Performing the standard procedure [37] of abelianizing the constraints (2.2.21) and building the analog of the $\Omega_{BRS}^{(F)}$ of Eq.(2.1.30), it is then easy to see that the physical state condition (2.2.25) is equivalent to the following four conditions:

$$\left\{ \begin{array}{l} \tilde{\mathcal{H}} | \text{phys} \rangle = 0 \\ Q_{(1)} | \text{phys} \rangle = 0 \\ \Pi_\alpha | \text{phys} \rangle = 0 \\ \Pi_g | \text{phys} \rangle = 0 \end{array} \right. \quad (2.2.27)$$

Let us now pause for a moment and, for completeness, let us briefly review the concept of equivariant cohomology (for references see [8]). Let us indicate with ψ and χ two *inhomogeneous* forms on a symplectic space and with V a vector field on the same space. One says that the form ψ is equivariantly closed but not exact with respect to the vector field V if the following conditions are satisfied:

$$\begin{cases} \mathcal{L}_V \psi = 0 \\ \mathcal{L}_V \chi = 0 \\ (d - \iota_V) \psi = 0 \\ \psi \neq (d - \iota_V) \chi \end{cases} \quad (2.2.28)$$

The forms ψ and χ have to be inhomogeneous because, while the exterior derivative d increases the degree of the form of one unit, the contraction with the vector field ι_V decreases it of one unit, so the third and fourth relations in Eq.(2.2.28) would never have a solution if ψ and χ were homogeneous in the form degree. From now on let us use the notation: $d_{eq} \equiv (d - \iota_V)$ and let us try to interpret the relations contained in Eq.(2.2.28). Restricting the forms to satisfy the first two relations contained in (2.2.28) and noting that $d_{eq}^2 = -\mathcal{L}_V$, we have that on this restricted space $[d_{eq}]^2 = 0$, and so d_{eq} acts as an exterior derivative. If we now consider the last two relations of (2.2.28) it is then clear that they define a cohomology problem for d_{eq} .

There are other more abstract definitions of equivariant cohomology [54] based on the *basic* cohomology of the Weil algebra associated to a Lie-algebra, but we will not dwell on it here. Equivariant cohomology is a concept that entered also the famous localization formula of Duistermaat and Heckman [23] thanks to the work of Atiyah and Bott, and into Topological Field Theory [61] thanks to the work of R. Stora and collaborators.

Let us now go back to our Lagrangian $\tilde{\mathcal{L}}_{eq}$ of Eq. (2.2.19) whose physical state space is restricted by the conditions (2.2.27) because of the gauge invariance given in (2.2.20). It is easy to realize that the first two physical state conditions of (2.2.27) are equivalent to the first and third conditions of Eq.(2.2.28) once the vector field V is identified with the Hamiltonian vector field $(dH)^\sharp$ (according to the notation of Ref.[1]). In fact let us remember the correspondence described in Section 1.3 between standard operations in differential geometry and in our formalism and in particular formula (1.3.28). This tells us that the Lie-derivative \mathcal{L}_{dH^\sharp} acts on a form as the commutator of $\tilde{\mathcal{H}}$ with the same form written in terms of c^a variables. This commutator gives the same result as the action of $\tilde{\mathcal{H}}$ on functions of φ^a and c^a once $\tilde{\mathcal{H}}$ is written as a differential operator like in (1.2.6)-(1.2.7). So Eq.(1.3.28) proves that the first condition in both Eqs.(2.2.28) and (2.2.27) is the same:

$$\mathcal{L}_{(dH^\sharp)} \psi = 0 \longrightarrow \tilde{\mathcal{H}} | \text{phys} \rangle = 0. \quad (2.2.29)$$

Next let us look at the second condition in (2.2.27) and the third in (2.2.28). From the form (2.2.16) of $Q_{(1)}$ we see that its first term, the Q_{BRS} , via the correspondence given by Eq.(1.3.18), corresponds to the exterior derivative d which is exactly the first term contained in the third relation of Eq.(2.2.28). The second term in $Q_{(1)}$ is the N_H which is given in Eq.(1.3.6) and can be written as

$$\overline{N}_H = [\overline{Q}_{BRS}, H] \quad (2.2.30)$$

From the relation (1.3.13) we see that we can interpret \overline{N}_H as the Hamiltonian vector field built out of the function H . Its action as a differential operator on forms is then given by Eq.(1.3.13), that means it acts as the interior contraction, $\iota_{(dH)^\sharp}$, of the Hamiltonian vector field with forms. This basically proves the correspondence between the second relation of (2.2.27) and the third of (2.2.28):

$$(d - \iota_{(dH)^\sharp})\psi = 0 \longrightarrow Q_{(1)} | \text{phys} \rangle = 0. \quad (2.2.31)$$

Note that this correspondence would not hold if we had gauged the Q_H , like we did in Section 2.1. In fact the Q_H , being made of Q_{BRS} and N_H and not \overline{N}_H , would not have had the meaning of equivariant exterior derivative.

Let us now conclude the proof that the conditions (2.2.27) are really equivalent to the equivariant cohomology problem given by Eq.(2.2.28). We have already explained in Eqs.(2.2.29) and (2.2.31) the correspondences¹²:

$$\begin{aligned} \mathcal{L}_V \psi = 0 &\longleftrightarrow \tilde{\mathcal{H}} | \text{phys} \rangle = 0 \\ (d - \iota_V)\psi = 0 &\longleftrightarrow Q_{(1)} | \text{phys} \rangle = 0. \end{aligned}$$

We have not discussed yet the 2nd and 4th equations in (2.2.28). These conditions are equivalent to the following statement:

$$\{\psi = (d - \iota_V)\chi \quad \text{with} \quad \mathcal{L}_V \chi = 0\} \implies \{\psi \simeq 0\}, \quad (2.2.32)$$

where the symbol \simeq means “cohomologically equivalent”. Therefore, if we want to complete the proof of the correspondence between the $|\text{phys}\rangle$ states of (2.2.27) and the ψ of (2.2.28), we must show that:

$$\{|\text{phys}\rangle = Q_{(1)}|\chi\rangle \quad \text{with} \quad \tilde{\mathcal{H}}|\chi\rangle = 0\} \implies \{|\text{phys}\rangle \simeq 0\}. \quad (2.2.33)$$

Note that the state $|\chi\rangle$ is not required to be physical, but only to satisfy the LHS of Eq.(2.2.32); this implies that $|\chi\rangle$ in general does not satisfy the third and the fourth conditions of (2.2.27). This means that $|\chi\rangle$ in general can depend on g and α . The point is that this dependence, due to the LHS of Eq. (2.2.33) and to the requirement that $|\text{phys}\rangle$ does not depend on g and α , must have the following form, as proven in Appendix A.6:

$$|\chi\rangle = |\chi_0\rangle + |\zeta; \alpha, g\rangle, \quad (2.2.34)$$

where $|\chi_0\rangle$ is independent of both g and α , and $|\zeta; \alpha, g\rangle \in \ker Q_{(1)}$. Moreover if $|\chi\rangle \in \ker \tilde{\mathcal{H}}$ (as imposed by Eq. (2.2.33)), also $|\chi_0\rangle \in \ker \tilde{\mathcal{H}}$ as one can check by

¹²In the following it is understood that $V = (dH)^\sharp$.

applying $Q_{(1)}^2 = -i\tilde{\mathcal{H}}$ to both members of Eq. (2.2.34). We are now ready to show that states of the form (2.2.33) are cohomologically equivalent to zero according to the cohomology defined by $\Omega_{BRS}^{(Eq.)}$ of Eq. (98). The proof goes as follows:

$$\begin{aligned}
|\text{phys}\rangle &= Q_{(1)} |\chi\rangle \\
&= Q_{(1)} |\chi_0\rangle \\
&= [C^{(1)}]^{-1} [C^{(1)} Q_{(1)} + C^H \tilde{\mathcal{H}} + \mathcal{P}_{(1)} \Pi_\alpha + \mathcal{P}_H \Pi_g + i(C^{(1)})^2 \overline{\mathcal{P}}_H] |\chi_0\rangle \\
&= [C^{(1)}]^{-1} \Omega_{BRS}^{eq} |\chi_0\rangle \\
&= \Omega_{BRS}^{(Eq.)} |\chi'\rangle;
\end{aligned} \tag{2.2.35}$$

(where we have defined $|\chi'\rangle \equiv [C^{(1)}]^{-1} |\chi_0\rangle$) and therefore $|\text{phys}\rangle$ is cohomologically equivalent to zero. Note that the ghost $C^{(1)}$ is bosonic in character and so we can build its inverse. In the third equality of (2.2.35) we have used the fact that the second, the third and the fourth terms give zero when applied to $|\chi_0\rangle$. The last term ($i(C^{(1)})^2 \overline{\mathcal{P}}_H$) also annihilates $|\chi_0\rangle$ by a similar reasoning based on the fact that $|\text{phys}\rangle$ cannot contain any dependence on C^H . This concludes our proof.

The next step would be to exploit the correspondence between Eqs.(2.2.27) and Eqs.(2.2.28) in the study of the topology of the space of classical trajectories. In fact, there is an important property of the equivariant forms: if the action of the group G (i.e. the group which defines the equivariant forms) on the manifold \mathcal{M} is free, that is:

$$g \cdot x = x \iff g = e \quad \forall x, \tag{2.2.36}$$

then we have the following isomorphism:

$$H_G(\mathcal{M}) \cong H(\mathcal{M}/G), \tag{2.2.37}$$

where we have indicated with H_G and H the equivariant and de-Rham cohomologies respectively. In our case, the RHS of the previous equation is the de-Rham cohomology of the quotient space $\mathcal{M}/\tilde{\mathcal{H}}$, which is precisely the space of the classical trajectories. This would mean that — in principle — we could study the geometry of the space of the classical trajectories by simply analyzing the physics of the model (2.2.19). Unfortunately the isomorphism (2.2.37) is not always true, because in the case of the Hamiltonian evolution the action of G on \mathcal{M} is not always free¹³. We hope to shed some further light on this topics in future works.

Basically with our path-integral we have managed to get the propagation of equivariantly non-trivial states by properly gauging the Susy. This is what the Susy is telling us from a geometrical point of view. It is not the first time that the equivariant cohomology is reduced to a sort of BRS formalism [47], but differently from these authors the BRS-BFV charge we obtained is really linked to a local invariance problem associated to the Lagrangian (2.2.19).

¹³For example in the harmonic oscillator the hamiltonian $H = \frac{1}{2}(p^2 + q^2)$ admits $p = q = 0$ as a fixed point.

2.3 Motion On Constant-Energy Surfaces.

In the previous sections we have gauged the Susy and we have ended up with a constrained motion on the hypersurfaces given by Eq.(2.1.20) or (2.2.21). In this section we shall reverse the procedure. We want to constrain the motion on some particular hypersurfaces of phase-space and see which local symmetries the associated Lagrangian will exhibit. The hypersurfaces we choose are those defined by fixed values of the constants of motion. We will explain later the reasons for this choice.

Let us start with the constant energy surface: $H(p, q) = E$. The most natural thing to do is to add this constraint to the $\tilde{\mathcal{L}}$ of Eq.(1.1.6):

$$\tilde{\mathcal{L}}_E \equiv \tilde{\mathcal{L}} + f(t)(H - E), \quad (2.3.1)$$

where $f(t)$ is a gauge variable. This Lagrangian has the following local invariance:

$$\begin{cases} \delta_H(\dots) = [\tau(t)H, (\dots)] \\ \delta_H f(t) = i\dot{\tau}(t), \end{cases} \quad (2.3.2)$$

where $\tau(t)$ is an infinitesimal bosonic parameter and we have indicated with (\dots) any of the variables $(\varphi^a, c^a, \lambda_a, \bar{c}_a)$. Anyhow this is not the whole story. In fact if we restrict the original phase-space to be a constant energy surface, the forms c^a themselves must be restricted to be those living only on the energy surface, that means they must be “perpendicular” to the gradient of the Hamiltonian. This constraint is:

$$c^a \partial_a H = 0 = N_H. \quad (2.3.3)$$

So basically we have to impose that the N_H -function of Eq.(1.3.6) be zero. This is a further constraint we should add to the $\tilde{\mathcal{L}}_E$ of Eq.(2.3.1). One may think that an analogous restriction has to be done also for the vector fields considering that forms and tensor fields are paired by the symplectic matrix as explained in Section 1.3. If we accept this we will have to add the condition that the \bar{N}_H of Eq.(1.3.7) be zero.

A manner to get all these constraints automatically, without having to add them by hand, is beautifully achieved if we request that the new Hamiltonian $\tilde{\mathcal{H}}_E$, which describes the motion on the energy surface, be a Lie-derivative along a Hamiltonian vector field like the original $\tilde{\mathcal{H}}$ was. $\tilde{\mathcal{H}}_E$ must be a Lie-derivative because, after all, the motion is the same as before. This time the difference is that we fix a particular value of the energy and so we include this initial condition directly into the Lagrangian. If $\tilde{\mathcal{H}}_E$ is a Lie-derivative along a Hamiltonian vector field then, from what we said in Chapter 1, we gather that $\tilde{\mathcal{H}}_E$ must be of the following form:

$$\tilde{\mathcal{H}}_E = [Q_{BRS}^E [\bar{Q}_{BRS}^E, (...)]], \quad (2.3.4)$$

that means it must be the BRS variation of the antiBRS variation of some function that we have indicated with (...). The BRS and antiBRS charges in (2.3.4) are not the ones relative to $\tilde{\mathcal{H}}$, that means those of Eqs.(1.3.1)(1.3.2). For this reason we have indicated them with different symbols.

If $\tilde{\mathcal{H}}_E$ is of the form above then the associated Lagrangian must be BRS invariant. Let us start from $\tilde{\mathcal{L}}_E$ in Eq.(2.3.1) and see if it is BRS invariant at least under the old Q_{BRS} . It is easy to do that calculation and we get:

$$[Q_{BRS}, \tilde{\mathcal{L}} + f(H - E)] = f(t)N_H \neq 0. \quad (2.3.5)$$

So it is not BRS invariant. The way out is to add to $\tilde{\mathcal{L}}_E$ the N_H multiplied by a gauge field. The new Lagrangian is:

$$\tilde{\mathcal{L}}'_E \equiv \tilde{\mathcal{L}} + f(t)(H - E) + i\alpha(t)N_H, \quad (2.3.6)$$

where $\alpha(t)$ is the Grassmannian gauge field. This Lagrangian is BRS-invariant provided that we define a proper BRS-variation also on the gauge-fields $\alpha(t)$ and $f(t)$. These proper BRS transformations are:

$$\begin{cases} \delta(...) = [\epsilon Q_{BRS}, (...)] \\ \delta\alpha = i\epsilon f \\ \delta f = 0, \end{cases} \quad (2.3.7)$$

where we have indicated with (...) any of the $(\varphi^a, c^a, \lambda_a, \bar{c}_a)$ and with Q_{BRS} the old BRS charge of Eq.(1.3.1).

Next let us notice that if the form of $\tilde{\mathcal{H}}_E$ is the one of Eq.(2.3.4) then the associated Lagrangian has to be also antiBRS-invariant. Let us check if this happens with the $\tilde{\mathcal{L}}'_E$ of Eq.(2.3.6):

$$[\bar{Q}_{BRS}, \tilde{\mathcal{L}}'_E] = f(t)\bar{N}_H - \alpha(t)\tilde{\mathcal{H}}. \quad (2.3.8)$$

So it is not antiBRS invariant and the way out is again to add to $\tilde{\mathcal{L}}'_E$ the generators appearing on the RHS of (2.3.8). The final Lagrangian is:

$$\tilde{\mathcal{L}}''_E \equiv \tilde{\mathcal{L}} + f(t)(H - E) + i\alpha(t)N_H + i\bar{\alpha}(t)\bar{N}_H - g(t)\tilde{\mathcal{H}} \quad (2.3.9)$$

where $(f, \alpha, \bar{\alpha}, g)$ are gauge-fields. So we see that the request that our $\tilde{\mathcal{H}}_E$ be a Lie-derivative (2.3.4) has automatically produced the constraints $N_H = 0$ and $\bar{N}_H = 0$ that otherwise we would have had to add by hand like we did in the reasoning leading to Eq.(2.3.3).

The Lagrangian $\tilde{\mathcal{L}}_E''$ is invariant under the following generalized BRS and antiBRS transformations:

$$\delta_{Q_{BRS}} \equiv \begin{cases} \delta(\dots) = [\epsilon Q_{BRS}, (\dots)] \\ \delta\alpha = i\epsilon f \\ \delta\bar{\alpha} = 0 \\ \delta f = 0 \\ \delta g = \epsilon\bar{\alpha} \end{cases} \quad \bar{\delta}_{\bar{Q}_{BRS}} \equiv \begin{cases} \bar{\delta}(\dots) = [\bar{\epsilon}\bar{Q}_{BRS}, (\dots)] \\ \bar{\delta}\alpha = 0 \\ \bar{\delta}\bar{\alpha} = i\bar{\epsilon}f \\ \bar{\delta}f = 0 \\ \bar{\delta}g = -\bar{\epsilon}\alpha. \end{cases} \quad (2.3.10)$$

It is straightforward to build the BRS-antiBRS charges which produce the variations indicated above. They are:

$$Q_{BRS}^E \equiv Q_{BRS} + if\Pi_\alpha + i\bar{\alpha}\Pi_g \quad (2.3.11)$$

$$\bar{Q}_{BRS}^E \equiv \bar{Q}_{BRS} + if\Pi_{\bar{\alpha}} - i\alpha\Pi_g, \quad (2.3.12)$$

where Π_α , $\Pi_{\bar{\alpha}}$, Π_g are the momenta conjugate to the variables $\alpha, \bar{\alpha}, g$ and their graded commutators are:

$$[\alpha, \Pi_\alpha] = [\bar{\alpha}, \Pi_{\bar{\alpha}}] = i[\Pi_g, g] = i[f, \Pi_f] = 1. \quad (2.3.13)$$

The new BRS and antiBRS charges are nilpotent, as BRS charges should be, and anticommute among themselves

$$(Q_{BRS}^E)^2 = (\bar{Q}_{BRS}^E)^2 = [Q_{BRS}^E, \bar{Q}_{BRS}^E] = 0. \quad (2.3.14)$$

Having obtained these charges it is then easy to prove that the $\tilde{\mathcal{H}}_E''$ associated to the $\tilde{\mathcal{L}}_E''$ of Eq.(2.3.9) has the form (2.3.4) with the (...) in (2.3.9) given by $-i(H + g(H - E))$, i.e:

$$\tilde{\mathcal{H}}_E'' = -i[Q_{BRS}^E[\bar{Q}_{BRS}^E, H + g(H - E)]]. \quad (2.3.15)$$

This shows, with respect to the $\tilde{\mathcal{H}}$ of Eq.(1.1.8), that the 0-form out of which the Hamiltonian vector field is built is not H but $H + g(H - E)$. This is natural in the sense that this 0-form feels the constraint $H - E = 0$.

The symplectic structure behind our construction can be made more manifest if we introduce the following notation:

$$\begin{cases} \varphi^A \equiv (\varphi^a; g, \Pi_f) \\ \lambda_A \equiv (\lambda_a; \Pi_g, f) \\ c^A \equiv (c^a; \bar{\alpha}, \Pi_\alpha) \\ \bar{c}_A \equiv (\bar{c}_a; \Pi_{\bar{\alpha}}, \alpha) \end{cases}, \quad (2.3.16)$$

where the index in capital letter $(\dots)^A$ runs from 1 to $2n+2$ while the one in small letter $(\dots)^a$ runs from 1 to $2n$ and it refers to the usual variables $(\varphi^a, c^a, \lambda_a, \bar{c}_a)$. Let us also introduce an enlarged symplectic matrix:

$$\omega^{AB} = \begin{pmatrix} \omega^{ab} & 0 \\ 0 & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{pmatrix} \quad (2.3.17)$$

and then, using the definitions (2.3.16) and (2.3.17), the BRS-antiBRS charges (2.3.11) and (2.3.12) can be written in the following compact form:

$$\begin{cases} Q_{BRS}^E = ic^A \lambda_A \\ \bar{Q}_{BRS}^E = i\bar{c}_A \omega^{AB} \lambda_B. \end{cases} \quad (2.3.18)$$

Note that this form resembles very much the one of the original BRS and antiBRS charges (1.3.1)(1.3.2). It is also straightforward to prove that the $\tilde{\mathcal{H}}_E''$ has an N=2 supersymmetry like the old $\tilde{\mathcal{H}}$. To build the Susy charges we should first construct the (N_H, \bar{N}_H) charges analogous to those in Eqs.(1.3.6) and (1.3.7). Replacing in (1.3.7) the symplectic matrix and the variables with those constructed respectively in (2.3.17) and (2.3.16), and the 0-form H with the 0-form $H + g(H - E)$ entering the $\tilde{\mathcal{H}}_E$, we get:

$$\begin{cases} N_H^E = c^A \partial_A (H + g(H - E)) \\ \bar{N}_H^E = \bar{c}_A \omega^{AB} \partial_B (H + g(H - E)). \end{cases} \quad (2.3.19)$$

The supersymmetry charges analogous to those in Eqs.(1.3.8) and (1.3.9) are then

$$\begin{cases} Q_H^E \equiv Q_{BRS}^E - \beta N_H^E \\ \bar{Q}_H^E \equiv \bar{Q}_{BRS}^E + \beta \bar{N}_H^E, \end{cases} \quad (2.3.20)$$

where β is a dimensional parameter like the one appearing in (2.1.1). It is then easy to check that:

$$[Q_H^E, \bar{Q}_H^E] = 2i\beta \tilde{\mathcal{H}}_E''. \quad (2.3.21)$$

Up to now we have found which are the global symmetries of our Lagrangian (2.3.9), but let us not forget that the goal of this section was to find out if, by imposing a constraint from outside like the one of being on a constant energy surface, we would get a Lagrangian with local symmetries. It is actually so and a first hint was given by the local symmetry of Eq.(2.3.2). The full set of local invariances of the Lagrangian $\tilde{\mathcal{L}}_E''$ of Eq.(2.3.9) is:

$$\begin{cases} \delta(\dots) = [\tau H + \bar{\eta} \bar{N}_H + \eta N_H + \epsilon \tilde{\mathcal{H}}, (\dots)] \\ \delta f = i\dot{\tau} \\ \delta \alpha = \dot{\eta} \\ \delta \bar{\alpha} = \dot{\bar{\eta}} \\ \delta g = -i\dot{\epsilon}, \end{cases} \quad (2.3.22)$$

where $(\tau, \eta, \bar{\eta}, \epsilon)$ are the local gauge-parameters depending on t , and with (\dots) we have indicated the variables $(\varphi^a, c^a, \lambda_a, \bar{c}_a)$.

The above local symmetry is not a local supersymmetry as in the previous sections but a different graded one whose generators are $(H, N, \bar{N}, \tilde{\mathcal{H}})$. While before, in section 2.1 and 2.2, the local symmetry was a clearly recognizable one but the constraints — being in the enlarged space $(\varphi^a, c^a, \lambda_a, \bar{c}_a)$ — were hard to visualize, here we have the inverse situation: the constraint (the constant energy one) is easy to visualize but not so much the local symmetries.

For a moment let us stop these formal considerations and let us check that the Hamiltonian in Eq.(2.3.15) is the correct one. The procedure we have followed here of constraining the motion on a constant energy surface can be applied also to any other constant of motion $I(\varphi)$. The result would be the following Hamiltonian:

$$\tilde{\mathcal{H}}_I'' = \tilde{\mathcal{H}} - f[I(\varphi) - k] - i\bar{\alpha}\bar{N}_{(I)} - i\alpha N_{(I)} + g\tilde{\mathcal{I}} \quad (2.3.23)$$

where k is a constant and

$$\begin{cases} N_{(I)} = c^a \partial_a I \\ \bar{N}_{(I)} = \bar{c}_a \omega^{ab} \partial_b(I) \\ \tilde{\mathcal{I}} = -i[Q_{BRS}, [\bar{Q}_{BRS}, I(\varphi)]] \end{cases} \quad (2.3.24)$$

If we had an integrable system with n constants of motion I_i in involution we would get as Hamiltonian the following expression:

$$\tilde{\mathcal{H}}_{int.}'' = \tilde{\mathcal{H}} - \sum_i \{f_i[I_i(\varphi) - k_i] - i\bar{\alpha}_i \bar{N}_{(I)_i} - i\alpha_i N_{(I)_i} + g\tilde{\mathcal{I}}_i\} \quad (2.3.25)$$

Let us now do a counting of the effective degrees of freedom of the Hamiltonian $\tilde{\mathcal{H}}_I''$ of Eq.(2.3.23). We have $8n$ variables $(\varphi^a, c^a, \lambda_a, \bar{c}_a)$, plus 4 gauge fields $(f, \alpha, \bar{\alpha}, g)$, plus 4 momenta associated to these gauge fields minus 4 primary constraints (which are the gauge-momenta equal zero), minus 4 secondary constraints $(I(\varphi) - k = 0, N_{(I)} = 0, \bar{N}_{(I)} = 0, \tilde{\mathcal{I}} = 0)$ minus 8 gauge-fixings for a total of $8n - 8$ independent phase-space variables. For the Hamiltonian of an integrable system like $\tilde{\mathcal{H}}_{int.}''$ this counting would give $8n - 8n = 0$ as effective number of phase-space variables describing the system. *This is absurd!* This situation could already be seen in the one-dimensional harmonic oscillator where $n = 1$ and we have just one constant of motion (the energy). The number of variables of the associated $\tilde{\mathcal{H}}_E''$ would be

$8n - 8 = 8 - 8 = 0$. One could claim that our $\tilde{\mathcal{H}}''_{int.}$, having zero degrees of freedom, actually describes a Topological-Field-Theory model, and maybe it is so but for sure it does not describe the motion taking place on the tori of an integrable system. On the tori we have the angles which vary with time but here, having effectively zero phase-space variables, we do not have any motion taking place at all. If it is a topological theory at most the $\tilde{\mathcal{H}}''_{int.}$ can describe some static *geometric* feature of our system. This in itself would be interesting and that is why we have carried this construction so far. We hope to come back to this issue in future papers but for the moment we want to go back from where we started, that is Eq.(2.3.1) and see which is the way to get a Hamiltonian describing really the motion on the constant energy surface.

What we basically want to get is a Hamiltonian whose counting of degrees of freedom is correct. At the basic phase-space level labelled by the variables φ we have $2n$ variables minus 1 constraint that is $H - E = 0$ so the total number is $2n - 1$. Going up to the space $(\varphi^a, c^a, \lambda_a, \bar{c}_a)$ this number should be multiplied by 4 that is $8n - 4$.

What went wrong in the construction of $\tilde{\mathcal{L}}''_E$ of Eq.(2.3.9)? One thing that we requested, but which was not necessary, was that the vector fields obey a constraint $\bar{N} = 0$ analogous to the one of the forms $N = 0$. We made that request only in order to maintain the standard pairing between tensor fields and forms which appear in any symplectic theory, but our theory is not a symplectic one anymore because the basic space in φ^a has odd dimension $2n - 1$ and cannot be a symplectic space. So let us release the request of having $\bar{N} = 0$. We could have a weaker request by adding this constraint via the derivative of a Lagrange multiplier (or gauge-field) in the same manner as we did in Eq.(2.2.11). There we realized that adding constraints in this manner does not decrease the number of degrees of freedom. By consistency then also the $\tilde{\mathcal{H}}$ constraint, which appeared together with the \bar{N} via the Eq.(2.3.8), should be added via the derivative of its associated Lagrange multiplier. So in order to describe the motion on constant energy surfaces, instead of (2.3.9) the Lagrangian we propose is:

$$L_E = \tilde{\mathcal{L}} + f(H - E) + i\dot{\alpha}\bar{N} + i\alpha N - \dot{g}\tilde{\mathcal{H}}. \quad (2.3.26)$$

The constraints (primary and secondary) are:

$$\left\{ \begin{array}{l} \Pi_f = 0 ; \quad H - E = 0 ; \\ \Pi_\alpha = 0 ; \quad N = 0 ; \\ \Pi_{\bar{\alpha}} = -i\bar{N} ; \\ \Pi_g = -\tilde{\mathcal{H}}. \end{array} \right. \quad (2.3.27)$$

They are 6, all first class, and we need 6 gauge-fixings. So doing now the counting of independent variables in phase-space we have: $8n$ variables $(\varphi^a, c^a, \lambda_a, \bar{c}_a)$, plus $4 + 4$ gauge-fields and their momenta, minus 6 constraints, minus 6 gauge fixings

for a total of $8n - 4$ which is exactly the number we wanted!

Let us analyze the difference between the last constraint in Eq.(2.3.27) (that is $\Pi_g = -\tilde{\mathcal{H}}$) and the one associated to the Lagrangian $\tilde{\mathcal{L}}_E''$ of Eq. (2.3.9) (that is $\tilde{\mathcal{H}} = 0$). This last constraint seems to totally freeze the motion while the one in Eq.(2.3.27) does not freeze it but just foliates the space of values of $\tilde{\mathcal{H}}$. Similar things can be said for the constraint $\bar{N} = 0$ associated to $\tilde{\mathcal{L}}_E''$ and the one, $\Pi_\alpha = -i\bar{N}$, associated to L_E . This last one would not force the vector fields in a configuration symplectically equivalent to the one of forms.

Let us now proceed to further analyze the Lagrangian L_E of Eq.(2.3.26). The associated Hamiltonian is:

$$H_E = \tilde{\mathcal{H}} + \Pi_\alpha \dot{\alpha} + \Pi_g \dot{g} + \Pi_f \dot{f} + \Pi_{\bar{\alpha}} \dot{\bar{\alpha}} - f(H - E) - i\alpha N - i\dot{\bar{\alpha}}\bar{N} + \dot{g}\tilde{\mathcal{H}}, \quad (2.3.28)$$

where we had to leave in some velocities because we could not perform the Legendre transform. From the above Hamiltonian we can go to the canonical one [22] by imposing the primary constraints. The result is:

$$H_E^{can.} \equiv \tilde{\mathcal{H}} - f(H - E) - i\alpha N. \quad (2.3.29)$$

It is easy to prove that this $H_E^{can.}$ is a Lie-derivative along a vector field but not along a Hamiltonian vector-field. To show that let us first define the following new variables:

$$\left\{ \begin{array}{l} \varphi^A = (\varphi^a, \pi_f) \\ \lambda_A = (\lambda_a, f) \\ c^A = (c^a, \Pi_\alpha) \\ \bar{c}_A = (\bar{c}_a, \alpha). \end{array} \right. \quad (2.3.30)$$

In this enlarged phase-space the BRS charge (or exterior derivative) is

$$Q_{BRS}^{can.} = Q_{BRS} + if\Pi_\alpha, \quad (2.3.31)$$

and the analog of the Hamiltonian vector field \bar{N}_H is

$$\bar{N}_H^{can.} = \bar{N}_H - \bar{\alpha}(H - E), \quad (2.3.32)$$

which is not a Hamiltonian vector field anymore because it cannot be written as the antiBRS variation of something as a Hamiltonian vector field should be (see Eq.(1.3.7)).

The proof that $H_E^{can.}$ of Eq.(2.3.29) is the Lie-derivative along the vector field $\bar{N}_H^{can.}$ above comes from the fact that it can be written as the commutator of that vector field with the exterior derivative $Q_{BRS}^{can.}$ above:

$$H_E^{can.} = -i[Q_{BRS}^{can.}, \bar{N}_H^{can.}]. \quad (2.3.33)$$

Proving this relation is straightforward. One just needs to use the standard commutators plus the following ones:

$$[\alpha, \Pi_\alpha] = 1; \quad [f, \Pi_f] = -i. \quad (2.3.34)$$

Eq.(2.3.33) implies that $H_E^{can.}$ of Eq.(2.3.29) is invariant under the global BRS transformations generated by the $Q_{BRS}^{can.}$ of Eq.(2.3.31). It is also easy to see that the Lagrangian L_E of Eq.(2.3.26) has the following local invariances different from those of Eq.(2.3.22):

$$\left\{ \begin{array}{l} \delta(\dots) = [\tau H + \bar{\eta} \bar{N}_H + \eta N_H + \epsilon \tilde{\mathcal{H}}, (\dots)] \\ \delta f = i\dot{\tau} \\ \delta \alpha = \dot{\eta} \\ \delta \bar{\alpha} = \bar{\eta} \\ \delta g = -i\epsilon. \end{array} \right. \quad (2.3.35)$$

Again, as before, this is a local symmetry but not a local supersymmetry.

Regarding the supersymmetry we can find a global one under which our $H_E^{can.}$ of Eq.(2.3.29) is invariant. It is the one generated by the following charge:

$$Q_{Susy} = Q_{BRS}^{can.} + \bar{N}_H^{can.}, \quad (2.3.36)$$

which is a Susy charge because it is easy to prove that:

$$[Q_{Susy}]^2 = iH_E^{can.}. \quad (2.3.37)$$

Differently from the $\tilde{\mathcal{H}}$ of our original system, we do not have an N=2 supersymmetry like in Eq.(2.3.4), but only an N=1 Susy. This is due to the loss of a symplectic structure on the constant energy surface.

The reason to work out this supersymmetry is not just academical. In fact we proved in Ref. [35] that there is a nice interplay between the loss of ergodicity of the system whose Hamiltonian is H and the spontaneous breaking of the Susy of $\tilde{\mathcal{H}}$. We proved in particular that if the Susy of $\tilde{\mathcal{H}}$ is unbroken then the system described by H is in the ergodic phase and that if the system is in the ordered phase (non-ergodic) then the Susy of $\tilde{\mathcal{H}}$ must be broken. We could not prove the inverse of these two statements that is that if the system is in the ergodic phase then the Susy must be unbroken and that if the Susy is broken then the system must be in an ordered or non-ergodic regime. The reason we could not prove these inverse statements was that the energy at which the motion took place had not been specified. We have no time here to explain the detailed reasons why this lack of specification could not allow us to do the inverse of that statement and we advice the reader interested in understanding this point to study in detail the full

set of papers contained in Ref. [35]. Ergodicity [5] is a concept which is strongly energy dependent: a system can be ergodic at some energy and not ergodic at other energies. So it was crucial to develop a formalism giving us the motion on constant energy surfaces like we have done here. The parameter E entering our $H_E^{can.}$ is not a phase-space variable and we can consider it as a coupling constant. We know that at some values of the coupling a symmetry can be broken while at others it can be restored. In (2.3.26) the term containing the energy is like a tadpole term because it is proportional to a term linear in the field (the field in this case is $f(t)$ while the coupling is E).

The attempt to have a formulation of the CPI in which E enters explicitly was tried before [55] but along a different route. In that paper E was not a coupling constant but a degree of freedom conjugate to time in a formulation of CM invariant under time-reparametrization. We think that, in order to understand the interplay *Susy/ergodicity*, it is better to treat E as a coupling constant.

The next step would be to check whether the Susy charge (2.3.36) we have in $H_E^{can.}$ is that for which the theorem [35] mentioned above, regarding the interplay *Susy/ergodicity*, holds also in the inverse form. If this were the case then we would have a criterion to check if a system (at some energy) is ergodic or not using a universal symmetry like Susy. Maybe even a sort of Witten index could be built which, by signaling if the Susy is broken or not, could tell us if the system is ergodic or not.

All this project will be left to future works because there are several other difficulties that have to be cleared before really embarking on a full understanding of the interplay between Susy and ergodicity. The main difficulty is the presence of zero and negative norm states which prevents the proper use of something like a Witten index for the study of the above mentioned interplay.

At present we are just working on the issue about how to give the CPI a fully satisfactory and consistent Hilbert space structure, with positive-definite inner product and unitary evolution (i.e. the choice of the scalar product must imply that $\tilde{\mathcal{H}}$ is hermitian). The results of this analysis will be contained in a forthcoming paper [20].

3. CPI and κ -symmetry

In the previous chapter we have analyzed some of the universal symmetries of the CPI Lagrangian (1.1.6). In particular we focused on the geometrical meaning of the classical supersymmetry generated by the two charges Q_H and \overline{Q}_H . In this chapter we switch our analysis to the two other fermionic charges D_H and \overline{D}_H introduced in Eqs.(1.4.11) and (1.4.12). These charges — as we have already claimed in Chapter 1 — are strictly related to Q_H and \overline{Q}_H because in superspace they become the covariant derivatives associated to these Susy charges.

Following the lines of Ref.[18], we make these two symmetries (D_H and \overline{D}_H) local and we note that the model we get exhibits a nonrelativistic local Susy which is very similar to the famous κ -symmetry introduced in the literature almost 20 years ago by Siegel [52], who discovered it in the Lagrangian of the supersymmetric relativistic particle without mass. This model was analyzed also in many following papers [16][46], where it was clarified that the most remarkable feature of this κ -symmetry is that it gives rise to some problems in separating 1st-class from 2nd-class constraints, and therefore in quantizing the model [53][41]. In our nonrelativistic framework, the main difference with respect to the relativistic model above is that no difficulty arises in the separation of 1st-class from 2nd-class constraints because in our case no 2nd-class constraint survives after imposing the invariance under local reparametrizations of time [17].

3.1 The κ -symmetry

The model studied by Siegel [52] for the massless relativistic superparticle is characterized by the following (1st order) action:

$$S = \int d\tau \left\{ p_\mu \left[\dot{x}^\mu - \frac{i}{2} \left(\overline{\zeta} \gamma^\mu \dot{\zeta} - \dot{\overline{\zeta}} \gamma^\mu \zeta \right) \right] - \frac{1}{2} \lambda p^2 \right\}, \quad (3.1.1)$$

where x^μ are n -dimensional space-time coordinates, ζ^a and $\overline{\zeta}_a$ are Dirac spinors and λ is a Lagrange multiplier introduced to realize the $p^2 = 0$ constraint. This

action is invariant under the following transformations:

τ -reparametrization (local)

$$\begin{aligned} \delta x^\mu &= \epsilon \dot{x}^\mu & \delta p_\mu &= \epsilon \dot{p}_\mu \\ \delta \zeta &= \epsilon \dot{\zeta} & \delta \bar{\zeta} &= \epsilon \dot{\bar{\zeta}} \\ \delta \lambda &= (\epsilon \dot{\lambda}) \end{aligned} \quad (3.1.2)$$

Supersymmetry (global)

$$\begin{aligned} \delta x^\mu &= \frac{i}{2} (\bar{\epsilon} \gamma^\mu \zeta - \bar{\zeta} \gamma^\mu \epsilon) & \delta p_\mu &= 0 \\ \delta \zeta &= \epsilon & \delta \bar{\zeta} &= \bar{\epsilon} \\ \delta \lambda &= 0 \end{aligned} \quad (3.1.3)$$

κ -symmetry (local)

$$\begin{aligned} \delta x^\mu &= \frac{i}{2} (\bar{\zeta} \gamma^\mu \not{x} \kappa - \bar{\kappa} \not{x} \gamma^\mu \zeta) & \delta p_\mu &= 0 \\ \delta \zeta &= \not{x} \kappa & \delta \bar{\zeta} &= \bar{\kappa} \not{x} \\ \delta \lambda &= 2i(\dot{\bar{\zeta}} \kappa - \bar{\kappa} \dot{\zeta}) \end{aligned} \quad (3.1.4)$$

where in (3.1.2) the dot means derivation with respect to τ and \not{x} is obviously $p_\mu \gamma^\mu$. As specified above, ϵ and $\kappa, \bar{\kappa}$ are local parameters (the first is a commuting scalar, the others are anticommuting spinors) while ϵ and $\bar{\epsilon}$ are two global (i.e. they do not depend on the base space τ) spinorial parameters. We are particularly interested in the structure of the third symmetry, which has been deeply analyzed in the literature. Here we want to give a pedagogical description of the structure of the transformation in phase space, and we want to highlight the role of the various operators and various commutation structures (Dirac Brackets) involved. This will turn out to be useful when we will analyze the analog of the κ -symmetry in Classical Mechanics.

First of all we notice that the first and third symmetries above are strictly related. In fact, if we release the $p^2 = 0$ constraint introducing a mass m in (3.1.1) we get

$$S_m = \int d\tau \left\{ p_\mu \left[\dot{x}^\mu - \frac{i}{2} (\bar{\zeta} \gamma^\mu \dot{\zeta} - \dot{\bar{\zeta}} \gamma^\mu \zeta) \right] - \frac{1}{2} \lambda (p^2 - m^2) \right\}. \quad (3.1.5)$$

S_m is still invariant under (3.1.3) but the other two invariances are lost. This is easy to see in phase space if we apply the Dirac procedure to the actions (3.1.1)

and (3.1.5). Consider first the massive model. The constraints are the following:

$$\text{First Class} \quad \begin{cases} \Pi_\lambda = 0 & (a) \\ p^2 - m^2 = 0 & (b) \end{cases} \quad (3.1.6)$$

$$\text{Second Class} \quad \begin{cases} \Pi_p^\mu = 0 & (a) \\ (\Pi_x)_\mu - p_\mu = 0 & (b) \\ D^a \equiv (\Pi_{\bar{\zeta}})^a + \frac{i}{2}(\not{x}\zeta)^a = 0 & (c) \\ \bar{D}_a \equiv (\Pi_\zeta)_a + \frac{i}{2}(\bar{\zeta}\not{x})_a = 0 & (d), \end{cases} \quad (3.1.7)$$

where $\Pi_{(\dots)}$ are the momenta conjugated¹⁴ to the variables indicated as (\dots) , which satisfy the following (graded) Poisson Brackets¹⁵:

$$\begin{aligned} [\lambda, \Pi_\lambda]_- &= 1; & [x^\mu, p_\nu]_- &= \delta_\nu^\mu; \\ [\zeta^a, (\Pi_\zeta)_b]_+ &= \delta_b^a; & [\bar{\zeta}_a, (\Pi_{\bar{\zeta}})^b]_+ &= \delta_a^b. \end{aligned} \quad (3.1.8)$$

The first thing to do is to construct the Dirac Brackets associated to the 2nd-class constraints. If we define the matrix

$$\Delta_{ij} = [\phi_i, \phi_j]_{PB} \quad (3.1.9)$$

where ϕ_k are the second class constraints, we have that the Dirac Brackets between two generic variables A, B of phase space are defined as:

$$[A, B]_{DB} = [A, B]_{PB} - [A, \phi_i]_{PB}(\Delta^{-1})^{ij}[\phi_j, B]_{PB}. \quad (3.1.10)$$

It is not difficult even if rather long (the details are confined in Appendix B) to find out that the Dirac Brackets at hand are:

$$[x^\mu, p_\nu]_{DB} = \delta_\nu^\mu; \quad (3.1.11)$$

$$[\zeta^a, (\Pi_\zeta)_b]_{DB} = \frac{1}{2}\delta_b^a; \quad [\bar{\zeta}_a, (\Pi_{\bar{\zeta}})^b]_{DB} = \frac{1}{2}\delta_a^b; \quad (3.1.12)$$

$$[\zeta^a, \bar{\zeta}_b]_{DB} = i(\not{x}^{-1})_b^a; \quad (3.1.13)$$

$$[x^\mu, \bar{\zeta}_b]_{DB} = -\frac{(\bar{\zeta}\gamma^\mu\not{x}^{-1})_b}{2}; \quad (3.1.14)$$

$$[\zeta^a, x^\mu]_{DB} = \frac{(\not{x}^{-1}\gamma^\mu\zeta)^a}{2}; \quad (3.1.15)$$

$$[x^\mu, (\Pi_\zeta)_b]_{DB} = -\frac{i}{4}(\bar{\zeta}\gamma^\mu)_b; \quad (3.1.16)$$

$$[x^\mu, (\Pi_{\bar{\zeta}})^a]_{DB} = -\frac{i}{4}(\gamma^\mu\zeta)^a \quad (3.1.17)$$

¹⁴Here and in the sequel we choose right derivatives for Grassmannian variables: $\Pi_\zeta := \frac{\overrightarrow{\partial} L}{\partial \zeta}$.

¹⁵In the sequel we shall omit the subscripts $+$ and $-$.

Once we have built the correct structure in phase space, it is not difficult to realize that the generators of the *global* supersymmetry are the following operators:

$$Q = \not{\epsilon} \zeta; \quad (3.1.18)$$

$$\overline{Q} = \overline{\zeta} \not{\epsilon}; \quad (3.1.19)$$

which reproduce precisely the transformations (3.1.3) if we define:

$$\delta(\dots) \equiv [(\dots), i\overline{\epsilon}Q - i\overline{Q}\epsilon]_{DB}. \quad (3.1.20)$$

Note that the minus sign in the RHS of the previous equation is chosen because of the anticommuting character of the parameter ϵ . Moreover we have:

$$[Q, \overline{Q}]_{DB} = i\not{\epsilon}, \quad (3.1.21)$$

which confirms that Q and \overline{Q} are two supersymmetry charges. Notice that we can induce the same Susy-transformations through the following operators:

$$Q' = i\Pi_{\overline{\zeta}} + \frac{1}{2}\not{\epsilon}\zeta; \quad (3.1.22)$$

$$\overline{Q}' = i\Pi_{\zeta} + \frac{1}{2}\overline{\zeta}\not{\epsilon}; \quad (3.1.23)$$

which is obvious because $Q \approx Q'$ and $\overline{Q} \approx \overline{Q}'$.

Let us now switch to the massless case (3.1.1). The main difference is that we cannot repeat all the steps of the previous analysis. In fact the new constraint $p^2 = 0$ implies that the matrix Δ of Eq.(3.1.9) is no longer invertible. This is due to the fact that $\det \Delta \propto \det(\not{\epsilon}) \propto (p^\mu p_\mu)^2 \approx 0$. Thus the construction of the Dirac Brackets is not as simple as in the massive case. In fact half of the constraints in Eq.(3.1.7) are now 1st-class while the other half remains 2nd-class and the separation of the two sets is not quite easy: see for example Refs.[41]. Nevertheless we can list the generators of the κ -transformations of Eq.(3.1.4):

$$K = i\not{\epsilon}D = i\not{\epsilon}\Pi_{\overline{\zeta}} - \frac{1}{2}\not{\epsilon}^2\zeta; \quad (3.1.24)$$

$$\overline{K} = i\overline{D}\not{\epsilon} = i\Pi_{\zeta}\not{\epsilon} - \frac{1}{2}\overline{\zeta}\not{\epsilon}^2 \quad (3.1.25)$$

(K and \overline{K} generate the transformation (3.1.4) through commutators like those in (3.1.20)). Obviously we should remember that (K, \overline{K}) are not a set of independent constraints because of the reason claimed above ($\not{\epsilon}$ is not invertible on the shell of the constraints). Note that we can write down the form of the generators K, \overline{K} even if we do not know exactly the form of the Dirac Brackets in this particular case. We can do that because the K, \overline{K} constraints commute (weakly) with all the constraints in (3.1.7c) and (3.1.7d) and therefore we have $[K, (\dots)]_{DB} \approx [K, (\dots)]_{PB}$ (and the same holds for \overline{K}) whatever are the surviving 2nd-class constraints determining the Dirac Brackets at hand.

3.2 κ -symmetry and CPI

In Chapter 1 we showed that the formalism of the Classical Path Integral exhibits a universal *global* Supersymmetry. However, differently from the model of Siegel, it does not possess any local invariance. If we want to build up a nonrelativistic analog of the model introduced in Section 1, we firstly must inject the local t -reparametrization invariance into the Lagrangian (1.1.6) by adding the corresponding constraint via a Lagrange multiplier g :

$$\tilde{\mathcal{L}}_1 \equiv \tilde{\mathcal{L}} + g\tilde{\mathcal{H}}. \quad (3.2.1)$$

In fact it is easy to see that the previous Lagrangian is *locally* invariant under

$$\begin{cases} \delta(\dots) = [(\dots), \epsilon(t)\tilde{\mathcal{H}}] \\ \delta g = -i\dot{\epsilon}(t); \end{cases} \quad (3.2.2)$$

where (\dots) denotes one of the variables $(\varphi^a, \lambda_b, c^a, \bar{c}_b)$. Moreover it is easy to check that it remains *globally* invariant under the $N = 2$ classical Susy of Eqs.(1.3.8) and (1.3.9). Nevertheless, in this simple model no local Susy is still present. If we want to complete the analogy, we must add (following the lines of Ref.[18]) two further constraints to the Lagrangian (3.2.1) and we get:

$$\tilde{\mathcal{L}}_2 \equiv \tilde{\mathcal{L}} + \xi D_H + \bar{\xi} \bar{D}_H + g\tilde{\mathcal{H}}. \quad (3.2.3)$$

In the previous equation D_H and \bar{D}_H are the operators introduced in Eqs.(1.4.11) and (1.4.12). We want to analyze this model following the same steps we used in Section 1 for the Lagrangian (3.1.1).

First of all we remember again that, in our non-relativistic case, the analog of the “ $p^2 = 0$ ” constraint is represented by the last term $g\tilde{\mathcal{H}}$, which produces the constraint $\tilde{\mathcal{H}} = 0$. Thus, as we did in Eq.(3.1.5), we start our analysis by releasing this constraint in the following way:

$$\tilde{\mathcal{L}}'_2 \equiv \tilde{\mathcal{L}} + \xi D_H + \bar{\xi} \bar{D}_H + g(\tilde{\mathcal{H}} - \tilde{E}), \quad (3.2.4)$$

which is the analog of Eq.(3.1.5). It should be remembered that \tilde{E} is not the energy of the system, but just a parameter related to the invariance under local time reparametrization: if $\tilde{E} = 0$ this symmetry is present, while if $\tilde{E} \neq 0$ this symmetry is lost.

One can immediately work out the constraints:

$$\text{First Class} \quad \begin{cases} \Pi_\xi = \Pi_{\bar{\xi}} = \Pi_g = 0; \\ \tilde{\mathcal{H}} - \tilde{E} = 0; \end{cases} \quad (3.2.5)$$

$$\text{Second Class} \quad \begin{cases} D_H = 0; \\ \bar{D}_H = 0, \end{cases} \quad (3.2.6)$$

Now we can compare the previous constraints with those in Eqs.(3.1.6) and (3.1.7). Concerning the 1st-class constraints, we notice that $\tilde{\mathcal{H}} - \tilde{E} = 0$ is the classical analog of the relativistic mass-shell constraint $p^\mu p_\mu - m^2 = 0$. This implies that $\Pi_g = 0$ plays the same role as $\Pi_\lambda = 0$ in the relativistic case, while the remaining two constraints ($\Pi_\zeta = 0$ and $\Pi_{\bar{\zeta}} = 0$) have no analog in the relativistic case. Consider now the 2nd-class constraints. The first thing to point out is that $D_H = 0$ and $\overline{D}_H = 0$ are precisely the classical analogs of $D^a = 0$ and $\overline{D}_b = 0$ in the relativistic case. We can say that because D_H and \overline{D}_H are related to the classical Susy charges Q_H and \overline{Q}_H in the same way in which D^a and \overline{D}_b are related to the relativistic Susy charges Q^a and \overline{Q}_b . In fact it is easy to see that in the relativistic framework D^a and \overline{D}_b commute with Q^a and \overline{Q}_b and $[D^a, \overline{D}_b] = [Q^a, \overline{Q}_b] = i\mathcal{P}^a_b$ in the same way in which, in the nonrelativistic context, D_H and \overline{D}_H commute with Q_H and \overline{Q}_H and $[D_H, \overline{D}_H] = [Q_H, \overline{Q}_H] = 2i\beta\tilde{\mathcal{H}}$. This is actually the kernel of the analogy. We start from a model which possesses a universal Susy generated by Q_H and \overline{Q}_H and we want to check whether it is possible to implement a classical analog of the relativistic κ -symmetry of Siegel. Since in the relativistic case the 2nd-class constraints are $D^a = 0$ and $\overline{D}_b = 0$, we modify the CPI-Lagrangian (1.1.6) in such a way that the resulting extension provides as 2nd-class constraints the classical analogs of D^a and \overline{D}_b , that is D_H and \overline{D}_H . This is precisely the model (3.2.4).

If we go on with the same steps as in Section 1 we find that the matrix $\Delta_{ij} = [\phi_i, \phi_j]$ has the form:

$$\Delta = \begin{pmatrix} 0 & 2i\beta\tilde{\mathcal{H}} \\ 2i\beta\tilde{\mathcal{H}} & 0 \end{pmatrix} \implies \Delta^{-1} = \begin{pmatrix} 0 & (2i\beta\tilde{\mathcal{H}})^{-1} \\ (2i\beta\tilde{\mathcal{H}})^{-1} & 0 \end{pmatrix} \quad (3.2.7)$$

and consequently the Dirac Brackets deriving from (3.2.6) have the form:

$$[A, B]_{DB} = [A, B] - [A, \overline{D}_H](2i\beta\tilde{\mathcal{H}})^{-1}[D_H, B] - [A, D_H](2i\beta\tilde{\mathcal{H}})^{-1}[\overline{D}_H, B]. \quad (3.2.8)$$

Now that we have the correct structure of our phase space we can proceed with the analogy with the relativistic case. First of all we can prove that the two supersymmetry charges Q_H and \overline{Q}_H introduced in Eqs.(1.3.8)(1.3.9) become weakly equal to the Q_{BRS} and \overline{Q}_{BRS} charges:

$$Q_H \approx 2Q_{BRS} = 2ic^a\lambda_a; \quad (3.2.9)$$

$$\overline{Q}_H \approx 2\overline{Q}_{BRS} = 2i\bar{c}_a\omega^{ab}\lambda_b; \quad (3.2.10)$$

and consequently:

$$[Q_{BRS}, \overline{Q}_{BRS}]_{DB} = \frac{1}{4}[Q_H, \overline{Q}_H]_{DB} = \frac{i\beta}{2}\tilde{\mathcal{H}}. \quad (3.2.11)$$

This shows that Q_H and \overline{Q}_H are the analogs¹⁶ of the charges Q' and \overline{Q}' of Eqs.(3.1.22) and (3.1.23) while the Q_{BRS} and \overline{Q}_{BRS} charges are analogous to the Q and \overline{Q} charges of Eqs.(3.1.18) and (3.1.19).

Consider now the case in which $\tilde{E} = 0$. We reduce to the Lagrangian (3.2.3) and we see that something happens which is similar to the mechanism of κ -symmetry discussed in Section 1. In fact in that case we saw that half of the 2nd-class constraints became 1st-class. Here, on the other side, we notice that both the 2nd-class constraints $D_H = \overline{D}_H = 0$ become 1st-class. In other words all the constraints in the model (3.2.3) are gauge constraints and contribute to restrict the space of the physical states. Therefore we see that in our nonrelativistic framework there is no difficulty in separating 1st-class from 2nd-class constraints (like in the relativistic case). This is simply due to the fact that no 2nd-class constraint remains after imposing the constraint $\tilde{\mathcal{H}} = 0$ (which is the classical analog of $p_\mu p^\mu = 0$). Proceeding with the analogy it is very easy to construct the classical analogs of K and \overline{K} of Eqs.(3.1.24)(3.1.25), that is the generators of the nonrelativistic κ -symmetry. They are simply (NR stands for “Non Relativistic”):

$$K_{NR} = \tilde{\mathcal{H}} D_H; \quad (3.2.12)$$

$$\overline{K}_{NR} = \tilde{\mathcal{H}} \overline{D}_H; \quad (3.2.13)$$

and the local transformations (under which the Lagrangian (3.2.3) is invariant) generated by K_{NR} and \overline{K}_{NR} are:

$$\begin{cases} \delta(\dots) = [(\dots), \varkappa(t) K_{NR} + \overline{\varkappa}(t) \overline{K}_{NR}] \\ \delta\xi = -2i\dot{\varkappa}\tilde{\mathcal{H}} \\ \delta\overline{\xi} = -2i\dot{\overline{\varkappa}}\tilde{\mathcal{H}} \\ \delta g = -2i\beta(\overline{\xi}\varkappa + \xi\overline{\varkappa})\tilde{\mathcal{H}}. \end{cases} \quad (3.2.14)$$

It is interesting to determine the physical states selected by the theory defined by Eq.(3.2.3). Since all the constraints are now 1st-class, we must impose them on the states as follows:

$$\Pi_\xi \rho(\varphi, c; t) = \Pi_{\overline{\xi}} \rho(\varphi, c; t) = \Pi_g \rho(\varphi, c; t) = 0; \quad (3.2.15)$$

$$D_H \rho(\varphi, c; t) = 0; \quad (3.2.16)$$

$$\overline{D}_H \rho(\varphi, c; t) = 0; \quad (3.2.17)$$

$$\tilde{\mathcal{H}} \rho(\varphi, c; t) = 0; \quad (3.2.18)$$

¹⁶This is not in contradiction with what we said few lines above, that is that Q_H and \overline{Q}_H are the nonrelativistic analogs of Q^a and \overline{Q}_b . In fact it should be remembered that on the shell of the constraints we have $Q \approx Q'$, $\overline{Q} \approx \overline{Q}'$ (in the relativistic case) and $Q_H \approx 2Q_{BRS}$, $\overline{Q}_H \approx 2\overline{Q}_{BRS}$ (in the nonrelativistic case).

and it is not difficult to obtain that the resulting (normalizable¹⁷) physical states have the following shape:

$$\rho(\varphi, c) \propto \exp[-\beta H(\varphi)]. \quad (3.2.19)$$

This is precisely the *Gibbs* distribution characterizing the *canonical* ensemble, provided we interpret the β constant of Eqs.(1.3.8)(1.3.9) as $(k_B T)^{-1}$, where T plays the role of the temperature at which the system is in equilibrium. In fact we should remember that up to now the dimensional parameter β introduced in Eqs.(1.3.8) and (1.3.9) has not been restricted by any constraint. It is a completely free parameter with a dimension of $(Energy)^{-1}$ which characterizes the particular $N = 2$ classical supersymmetry. The canonical Gibbs state made its appearance earlier in the context of the CPI and precisely in the first of Refs.[35]. There it was shown that, in the pure CPI model (1.1.6), the zero eigenstates of $\tilde{\mathcal{H}}$ which are also Susy-invariant are precisely the canonical Gibbs states. In our model instead we have obtained the Gibbs states as the entire set of physical states associated to the gauge theory described by the Lagrangian (3.2.3).

However the model (3.2.3), though interesting for the peculiar physical subspace it determines, is not the nonrelativistic Lagrangian which is closest to the Siegel model. We mean that one should remember that the Lagrangian (3.2.3) gives rise to a canonical Hamiltonian of the form:

$$\tilde{\mathcal{H}}_2 \equiv \tilde{\mathcal{H}} - \xi D_H - \bar{\xi} \bar{D}_H - g \tilde{\mathcal{H}}, \quad (3.2.20)$$

but on the other hand we have already checked that the two couples of operators (Q_H, \bar{Q}_H) and (D_H, \bar{D}_H) close on $\tilde{\mathcal{H}}$ and not on $\tilde{\mathcal{H}}_2$. Therefore, if we want to construct a more precise nonrelativistic analog of the model of Siegel, we should consider a slightly modified version of the Lagrangian (3.2.3) which is:

$$\tilde{\mathcal{L}}_3 \equiv \tilde{\mathcal{L}} + \dot{\xi} D_H + \dot{\bar{\xi}} \bar{D}_H + g \tilde{\mathcal{H}}. \quad (3.2.21)$$

One can easily check that the Lagrangian (3.2.21) yields, a part from a factor $(1 - g)$, the same Hamiltonian as the CPI. Therefore we can proceed following the same steps as before: we turn the $\tilde{\mathcal{H}} = 0$ constraint into $\tilde{\mathcal{H}} - \tilde{E} = 0$

$$\tilde{\mathcal{L}}'_3 \equiv \tilde{\mathcal{L}} + \dot{\xi} D_H + \dot{\bar{\xi}} \bar{D}_H + g(\tilde{\mathcal{H}} - \tilde{E}) \quad (3.2.22)$$

and we find out that the new constraints are:

$$\begin{array}{ll} \text{1}^{\text{st}}\text{-Class} & \left\{ \begin{array}{l} \Pi_g = 0; \\ \tilde{\mathcal{H}} - \tilde{E} = 0; \end{array} \right. \quad \text{2}^{\text{nd}}\text{-Class} & \left\{ \begin{array}{l} \Pi_\xi + D_H \equiv D'_H = 0; \\ \Pi_{\bar{\xi}} + \bar{D}_H \equiv D'_H = 0. \end{array} \right. \end{array} \quad (3.2.23)$$

¹⁷Also a state of the form $\rho(\varphi, c) \propto \exp[\beta H(\varphi)] c^1 c^2 \dots c^{2n}$ would be admissible, but it is not normalizable in φ .

Then, it is easy to check that we can repeat all the considerations we did below Eq.(3.2.6), if we replace D_H and \overline{D}_H with D'_H and \overline{D}'_H . As a second remark, we notice that the two constraints $\Pi_\xi = \Pi_{\bar{\xi}} = 0$, which had no analog in the relativistic context, have now disappeared. Moreover, because $[D'_H, \overline{D}'_H] = [D_H, \overline{D}_H] = 2i\beta\tilde{\mathcal{H}}$, we have also that the Dirac Brackets remain the same as those in Eq.(3.2.8), which lead to Eqs.(3.2.9)-(3.2.11). Again, when we put $\tilde{E} = 0$, we obtain that the two 2nd-class constraints $D'_H = \overline{D}'_H = 0$ become both 1st-class, differently from the relativistic case. However, the two models described by the two Lagrangians (3.2.3) and (3.2.21) are not equivalent. There are basically two differences. The first is the new form of the nonrelativistic κ -symmetry which now reads:

$$\begin{cases} \delta(\dots) = \tilde{\mathcal{H}}[(\dots), \varkappa(t)D_H + \overline{\varkappa}(t)\overline{D}_H] \approx [(\dots), \varkappa(t)K'_{NR} + \overline{\varkappa}(t)\overline{K}'_{NR}] \\ \delta\xi = [\xi, \varkappa(t)K'_{NR} + \overline{\varkappa}(t)\overline{K}'_{NR}] = -i\varkappa\tilde{\mathcal{H}} \\ \delta\bar{\xi} = [\bar{\xi}, \varkappa(t)K'_{NR} + \overline{\varkappa}(t)\overline{K}'_{NR}] = -i\overline{\varkappa}\tilde{\mathcal{H}} \\ \delta g = 2i\beta\tilde{\mathcal{H}}(\dot{\bar{\xi}}\varkappa + \dot{\xi}\overline{\varkappa}). \end{cases} \quad (3.2.24)$$

where “ \approx ” is understood in the Dirac sense and

$$K'_{NR} \equiv \tilde{\mathcal{H}}D'_H; \quad \overline{K}'_{NR} \equiv \tilde{\mathcal{H}}\overline{D}'_H. \quad (3.2.25)$$

The second difference, which is the most important, is represented by the two physical spaces associated to the two models (3.2.3) and (3.2.21). In fact we have already seen that the physical states associated to the first model are the Gibbs distributions $\rho(\varphi) \propto \exp(-\beta H(\varphi))$; on the other hand the physical states determined by the Lagrangian (3.2.21) must obey the following conditions:

$$\Pi_g \rho(\varphi, c, \xi, \bar{\xi}, g) = 0; \quad D'_H \rho(\varphi, c, \xi, \bar{\xi}, g) = (-i\partial_\xi + D_H)\rho(\varphi, c, \xi, \bar{\xi}, g) = 0; \quad (3.2.26)$$

$$\tilde{\mathcal{H}} \rho(\varphi, c, \xi, \bar{\xi}, g) = 0; \quad \overline{D}'_H \rho(\varphi, c, \xi, \bar{\xi}, g) = (-i\partial_{\bar{\xi}} + \overline{D}_H)\rho(\varphi, c, \xi, \bar{\xi}, g) = 0. \quad (3.2.27)$$

It is not difficult to realize that the solution of Eqs.(3.2.26)-(3.2.27) has the form:

$$\rho(\varphi, c, \xi, \bar{\xi}, g) \propto \exp(-i\xi D_H - i\bar{\xi} \overline{D}_H) \tilde{\rho}(\varphi, c), \quad (3.2.28)$$

where

$$\tilde{\mathcal{H}} \tilde{\rho}(\varphi, c) = 0, \quad (3.2.29)$$

which implies that $\tilde{\rho}(\varphi, c)$ is a function of constants of motion only. Therefore we can say that the physical states associated to the Lagrangian (3.2.21) are isomorphic to the functions $\tilde{\rho}(\varphi, c)$ which are annihilated by the Hamiltonian $\tilde{\mathcal{H}}$ and are consequently constants of motion. Obviously the Gibbs distributions are a subset of them. This allows us to claim that the model (3.2.21) is actually more general

than that characterized by the Lagrangian (3.2.3). More precisely the theory described by (3.2.21) is equivalent to that characterized by the Lagrangian (3.2.1). In fact it is easy to see that the physical Hilbert space associated to the latter is characterized by the distributions $\tilde{\rho}(\varphi, c, g)$ obeying to the constraints:

$$\frac{\partial}{\partial g} \tilde{\rho}(\varphi, c, g) = 0; \quad \tilde{\mathcal{H}} \tilde{\rho}(\varphi, c, g) = 0; \quad (3.2.30)$$

and the physical space is precisely the same as that in (3.2.29), which is isomorphic to that determined by Eqs.(3.2.26)-(3.2.27).

4. The Rescaling of the Action

In the previous chapters we studied the geometrical meaning of some of the symmetry charges listed in Section 1.3. Here we want to continue this project, and we focus on a very particular symmetry which we have not introduced yet. The reason we introduce it only now is that this new transformation does not possess a canonical generator like all the other symmetries introduced in Section 1.3. This is due to the fact that the effect of the former is a rescaling of the action \tilde{S} of the CPI, which is not a canonical transformation. Nevertheless, if we go to the superspace $(t, \theta, \bar{\theta})$, we see that we can find the expression of the generator, which we shall call \mathcal{Q}_S , through which we can reconstruct the transformation on all the variables $(\varphi^a, c^a, \lambda_a, \bar{c}_a)$.

4.1 The symmetry charge \mathcal{Q}_S

First of all let us come back to the superspace representation of the charges (1.4.3)-(1.4.8):

$$\mathcal{Q}_{BRS} = -\frac{\partial}{\partial\theta}; \quad \bar{\mathcal{Q}}_{BRS} = \frac{\partial}{\partial\bar{\theta}}; \quad (4.1.1)$$

$$\mathcal{K} = \bar{\theta} \frac{\partial}{\partial\theta}; \quad \bar{\mathcal{K}} = \theta \frac{\partial}{\partial\bar{\theta}}; \quad (4.1.2)$$

$$\mathcal{Q}_g = \bar{\theta} \frac{\partial}{\partial\theta} - \theta \frac{\partial}{\partial\bar{\theta}}; \quad (4.1.3)$$

$$\mathcal{N}_H = \bar{\theta} \frac{\partial}{\partial t}; \quad \bar{\mathcal{N}}_H = \theta \frac{\partial}{\partial t}; \quad (4.1.4)$$

$$\mathcal{Q}_H = -\frac{\partial}{\partial\theta} - \bar{\theta} \frac{\partial}{\partial t}; \quad \bar{\mathcal{Q}}_H = \frac{\partial}{\partial\bar{\theta}} + \theta \frac{\partial}{\partial t}; \quad (4.1.5)$$

$$\tilde{\mathcal{H}} = i \frac{\partial}{\partial t}. \quad (4.1.6)$$

It is easy to see that in this framework all the representations of the various charges commute with \mathcal{H} , which is the superspace representation of the Hamiltonian $\tilde{\mathcal{H}}$.

This is obvious because they are symmetries of the CPI formalism and they commute with $\widetilde{\mathcal{H}}$ already at the level of the phase space $\widetilde{\mathcal{M}}$, where they are expressed as functions of $(\varphi^a, c^a, \lambda_a, \bar{c}_a)$ and the commutators are those defined in Eq.(1.2.3). However, at the level of superspace, it is not difficult to check that also the following operator

$$\mathcal{Q}_s = \theta \frac{\partial}{\partial \theta} + \bar{\theta} \frac{\partial}{\partial \bar{\theta}} \quad (4.1.7)$$

does commute with \mathcal{H} . In this sense we can say that (4.1.7) generates a symmetry of the system at hand. The main difference with respect to all the other operators listed in Eqs.(4.1.1)-(4.1.6) is that \mathcal{Q}_s does not correspond to any operator in $\widetilde{\mathcal{M}}$, and this is the reason why we did not introduce it together with all the others. The fact that \mathcal{Q}_s does not correspond to any operator in $\widetilde{\mathcal{M}}$ is strictly bound to the fact that it does not generate a *canonical* transformation on the variables $(\varphi^a, c^a, \lambda_a, \bar{c}_a)$. In fact, remember that the representation of an operator O in superspace is defined by:

$$\delta \Phi^a = [\epsilon O, \Phi^a] \equiv -\epsilon \mathcal{O} \Phi^a, \quad (4.1.8)$$

where the brackets $[..., ...]$ are the (canonical) commutators introduced either in Eq.(1.2.1) or in Eq.(1.2.3). Consequently, if we are given an operator \mathcal{O} and we can find its counterpart O , it also means that \mathcal{O} generates a canonical transformation on $\widetilde{\mathcal{M}}$. Actually this is not the case of \mathcal{Q}_s . In fact, let us substitute Eq.(4.1.7) into Eq.(4.1.8) and see what variation it induces on the variables $(\varphi^a, c^a, \lambda_a, \bar{c}_a)$:

$$\delta_{\mathcal{Q}_s} \Phi^a = -\epsilon \mathcal{Q}_s \Phi^a = -\epsilon (\theta c^a + \bar{\theta} \omega^{ab} \bar{c}_b + 2i\bar{\theta} \theta \omega^{ab} \lambda_b). \quad (4.1.9)$$

Comparing the coefficients with the same power of θ and $\bar{\theta}$ on both sides of the equation above, we get

$$\delta_{\mathcal{Q}_s} \varphi^a = 0 \quad (4.1.10)$$

$$\delta_{\mathcal{Q}_s} c^a = -\epsilon c^a \quad (4.1.11)$$

$$\delta_{\mathcal{Q}_s} \bar{c}_a = -\epsilon \bar{c}_a \quad (4.1.12)$$

$$\delta_{\mathcal{Q}_s} \lambda_a = -2\epsilon \lambda_a. \quad (4.1.13)$$

It is easy to check how the CPI-Lagrangian (see Eq.(1.1.6)) changes under the variations above:

$$\delta_{\mathcal{Q}_s} \widetilde{\mathcal{L}} = -2\epsilon \widetilde{\mathcal{L}}. \quad (4.1.14)$$

We can conclude that the transformations induced by \mathcal{Q}_s on the variables $(\varphi^a, c^a, \lambda_a, \bar{c}_a)$ are a symmetry of our system. In fact they just rescale the overall

Lagrangian $\tilde{\mathcal{L}}$ and therefore they keep the equations of motion invariant and these qualifies them as symmetry transformations. Of course they are *non-canonical* symmetries, as we have already claimed, because the rescaling of the whole Lagrangian is not a canonical transformation in $\tilde{\mathcal{M}}$.

We could ask whether this is a symmetry also of the Susy-QM model of Witten [59] or at least of the conformal-QM of Ref.[26] (which will be introduced in the following chapter). Interestingly enough, the answer is *No!*, The technical reason being that in those QM models the analogs of the λ_a variables enter the Lagrangian with a quadratic term while in our $\tilde{\mathcal{L}}$ they enter linearly. There is also an important physical reason why that symmetry is not present in those QM models while it is present in our CM one. The reason is that in QM one cannot rescale the action (as our symmetry does) due to the presence of \hbar setting a scale for it, while this can be done in CM where no scale is set. The reader may object that our transformation rescales the Lagrangian but not the action

$$\tilde{S} = \int \tilde{\mathcal{L}} dt \quad (4.1.15)$$

because one could compensate the rescaling of the $\tilde{\mathcal{L}}$ with a rescaling of t . That is not so because our \mathcal{Q}_s transforms only the Grassmannian partners of time $(\theta, \bar{\theta})$ and not the time itself.

Therefore, this generalized symmetry seems to be very meaningful in the transition from Classical to Quantum Mechanics: in CM it is a true symmetry, in QM it is lost. We hope to shed some further light on this issue in the future.

For the time being, let us continue our analysis about rescaling the action, but from a slightly different point of view. Up to now we have dealt with a noncanonical rescaling of the Lagrangian $\tilde{\mathcal{L}}$. If we look at the explicit form (1.1.6) of this Lagrangian, we notice that the transformation (4.1.14) can be induced by a rescaling of $H(\varphi)$ (or equivalently of $L(q, \dot{q})$) at the level of the ordinary phase space \mathcal{M} . Then, the question which spontaneously arises is whether we can think of the \mathcal{Q}_s -transformation as the extension of another symmetry of CM, formulated at the level of \mathcal{M} . This is the question we shall try to answer in the next sections. In other words we want to check whether there exists a *universal* transformation (because the \mathcal{Q}_s -rescaling is also universal) in the space (q, t) whose effect is the rescaling of the action of any classical system.

4.2 The MSA transformation

Consider a classical system defined by a set of coordinates (in the configurations space) denoted by $\{q(t)\}$, where $q(t)$ is an N -dimensional vector and $2N$ is the

dimension of the phase space. The action of the system is obviously:

$$S(t, t_0)_{[q(\tau), \dot{q}(\tau)]} = \int_{t_0}^t d\tau L(q(\tau), \dot{q}(\tau); \tau); \quad (4.2.1)$$

where the LHS means that S is an ordinary function of the endpoints (t, t_0) and a functional of the trajectories $\{q(\tau)\}$. In the following lines we want to find out a *universal* transformation of the time t which has the effect of rescaling the action (4.2.1) by a factor $(1 + \epsilon)$ (we consider for simplicity an infinitesimal rescaling) regardless of the form of the Lagrangian $L(q(\tau), \dot{q}(\tau), \tau)$ out of which the action in question is built. We make a further requirement and we assume that the configuration $q(t)$ be a scalar under the transformation we are looking for. This means that we are in search for a transformation

$$t' = f(t, t_0)_{[q(\tau), \dot{q}(\tau)]} = t + \delta t(t, t_0)_{[q(\tau), \dot{q}(\tau)]} \quad (4.2.2)$$

with the following properties:

$$\begin{cases} \int_{t'_0}^{t'} d\tau' L(q'(\tau'), \dot{q}'(\tau'), \tau') = (1 + \epsilon) \int_{t'_0}^{t'} d\tau' L(q'(\tau), \dot{q}'(\tau'); \tau'); \\ q'(\tau') = q(\tau). \end{cases} \quad (4.2.3)$$

From now on we shall use the following notation: we denote by δq and $\bar{\delta} q$ the following quantities:

$$\begin{cases} \delta q(\tau) \equiv q'(\tau') - q(\tau); \\ \bar{\delta} q(\tau) \equiv q'(\tau) - q(\tau); \end{cases} \quad (4.2.4)$$

the relation between δq and $\bar{\delta} q$ is given by:

$$\bar{\delta} q(\tau) = -\dot{q}'(\tau) \delta \tau + \delta q(\tau) \quad (4.2.5)$$

which, by use of Eq.(4.2.3), reduces to:

$$\bar{\delta} q(\tau) = -\dot{q}'(\tau) \delta \tau. \quad (4.2.6)$$

Finally the following variations will be useful:

$$\begin{cases} \delta \dot{q}(\tau) = \frac{d}{d\tau'} q'(\tau') - \frac{d}{d\tau} q(\tau) = -\dot{q}(\tau) (\delta \tau); \\ \bar{\delta} \dot{q}(\tau) = \frac{d}{d\tau} q'(\tau) - \frac{d}{d\tau} q(\tau) = \frac{d}{d\tau} \bar{\delta} q(\tau). \end{cases} \quad (4.2.7)$$

Now we can proceed and try to work out the transformation (4.2.2). First of all we can rewrite the first equation in (4.2.3) in the following way¹⁸:

$$\begin{aligned} \int_{t'_0}^{t'} d\tau' L'(q'(\tau'), \dot{q}'(\tau'), \tau') &= \int_{t'_0}^{t'} d\tau' L'(q(\tau), \dot{q}(\tau) + \delta\dot{q}(\tau), \tau + \delta\tau) = \\ &= \int_{t'_0}^{t'} d\tau' \left[L'(q(\tau), \dot{q}(\tau), \tau) + \frac{\partial L'}{\partial \dot{q}} \delta\dot{q} + \frac{\partial L'}{\partial \tau} \delta\tau \right]. \end{aligned} \quad (4.2.8)$$

But now we can use the definition of transformed action $S \rightarrow S'$:

$$S' = \int_{t'_0}^{t'} d\tau' L'(q'(\tau'), \dot{q}'(\tau'), \tau') = \int_{t_0}^t d\tau L(q(\tau), \dot{q}(\tau), \tau) = S \quad (4.2.9)$$

and rewrite Eq.(4.2.8) as follows:

$$\int_{t_0}^t d\tau L(q(\tau), \dot{q}(\tau), \tau) = \int_{t'_0}^{t'} d\tau' \left[L'(q(\tau), \dot{q}(\tau), \tau) + \frac{\partial L'}{\partial \dot{q}} \delta\dot{q} + \frac{\partial L'}{\partial \tau} \delta\tau \right]. \quad (4.2.10)$$

According to Eq.(4.2.3) we have that $L' = (1 + \epsilon)L$ and consequently we can write:

$$\int_{t_0}^t d\tau L(q(\tau), \dot{q}(\tau), \tau) = (1 + \epsilon) \int_{t'_0}^{t'} d\tau' \left[L(q(\tau), \dot{q}(\tau), \tau) + \frac{\partial L}{\partial \dot{q}} \delta\dot{q} + \frac{\partial L}{\partial \tau} \delta\tau \right] \quad (4.2.11)$$

which, since ϵ is an infinitesimal parameter, implies:

$$\int_{t_0}^t d\tau \frac{\partial \tau'}{\partial \tau} \left[L(q(\tau), \dot{q}(\tau), \tau) + \frac{\partial L}{\partial \dot{q}} \delta\dot{q} + \frac{\partial L}{\partial \tau} \delta\tau \right] = (1 - \epsilon) \int_{t_0}^t d\tau L(q(\tau), \dot{q}(\tau), \tau). \quad (4.2.12)$$

But remember that:

$$\frac{d\tau'}{d\tau} = 1 + \frac{d}{d\tau} \delta\tau, \quad (4.2.13)$$

and if we plug this result into Eq.(4.2.12) and keep only 1st-order terms, we obtain:

$$\int_{t_0}^t d\tau \frac{\partial L}{\partial \tau} \delta\tau - \int_{t_0}^t d\tau \frac{\partial L}{\partial \dot{q}} \dot{q} (\delta\tau) + \int_{t_0}^t d\tau (\delta\tau) L = -\epsilon \int_{t_0}^t d\tau L. \quad (4.2.14)$$

Suppose for simplicity that $\frac{\partial L}{\partial \tau} = 0$; then we can rewrite the previous equation as:

$$\int_{t_0}^t d\tau \frac{d}{d\tau} (\delta\tau) L - \int_{t_0}^t d\tau \frac{\partial L}{\partial \dot{q}} \dot{q} \frac{d}{d\tau} (\delta\tau) = -\epsilon \int_{t_0}^t d\tau L. \quad (4.2.15)$$

¹⁸Remember that we have chosen $\delta q(\tau) = 0$ (See Eq.(4.2.3)).

The previous equation must hold for every choice of (t_0, t) and therefore we arrive at the following expression:

$$\frac{d}{dt}(\delta t) = -\epsilon \frac{L(q(t), \dot{q}(t))}{L(q(t), \dot{q}(t)) - \dot{q}(t) \frac{\partial L}{\partial \dot{q}(t)}}. \quad (4.2.16)$$

If we denote by $H(q, \dot{q})$ the so-called “energy function” [30]:

$$H(q(t), \dot{q}(t)) \equiv -L(q(t), \dot{q}(t)) + \dot{q}(t) \frac{\partial L}{\partial \dot{q}(t)}, \quad (4.2.17)$$

we can rewrite Eq.(4.2.16) as

$$\frac{d}{dt}(\delta t) = \epsilon \frac{L(q(t), \dot{q}(t))}{H(q(t), \dot{q}(t))} \quad (4.2.18)$$

which finally yields:

$$\boxed{\delta t = \epsilon \int_0^t d\tau \frac{L(q(\tau), \dot{q}(\tau))}{H(q(\tau), \dot{q}(\tau))}}. \quad (4.2.19)$$

This is precisely the transformation we were looking for. For the moment we consider ϵ as a global parameter and this is the reason why we kept it outside the integral in (4.2.19). As we said in the Introduction, the name MSA is the acronym of “Mechanical Similarity Anomaly”. The name “Mechanical Similarity” was used by Landau [44] to denote a transformation which — though in a completely different context — had the effect of rescaling the classical action. The reason for the word “Anomaly”, on the other hand, will be clear in the next Section.

4.3 MSA as a “standard” symmetry

In the previous section we have built a general time-transformation which induces a rescaling of the overall action of *every* classical system, regardless of the form of the Lagrangian defining the system. Obviously this transformation is very peculiar; for example it is a *path-dependent* transformation, and we shall come back on that later. The reason why we think it is worth analyzing this “strange” symmetry is that we feel that its behaviour can become very interesting when we pass to the quantum domain. In particular our goal would be to implement this transformation in the Quantum Path Integral via a procedure à la Fujikawa [27]. What we could get is that this *universal classical* symmetry becomes *anomalous* at the quantum level (that is where the “Anomaly” word in MSA comes from) and in this case

it could be very interesting to evaluate this anomaly. Therefore, if we want to proceed along these lines, the first step we should do is to find what is the conserved current associated to this transformation. The point is that, as it stands, this is not a “standard” symmetry because it does not leave the action invariant, but rescales it. In this section we shall try to give the MSA a formulation in which it appears as a “standard” symmetry. The proposal we make is to enlarge the configuration space with two further variables (γ, S) and consider a new kind of *extended* Lagrangian L_{ext} defined as follows:

$$L_{ext}(q, \dot{q}, \gamma, S) \equiv L(q, \dot{q}) + \gamma(L(q, \dot{q}) - \dot{S}). \quad (4.3.1)$$

Next let us extend the transformation (4.2.19) to the new enlarged configuration space in the following way:

$$\begin{cases} \delta t = \epsilon \int_0^t d\tau \frac{L(q(\tau), \dot{q}(\tau))}{H(q(\tau), \dot{q}(\tau))} \\ \delta q = 0 \\ \delta S = ? \\ \delta \gamma = ? \end{cases} \quad (4.3.2)$$

and we want to find out which are the variations δS and $\delta \gamma$ turning the transformation (4.3.2) into a symmetry of the action associated to the Lagrangian L_{ext} in Eq.(4.3.1). Clearly, in order to be a symmetry of the latter, the variations in (4.3.2) must obey the following relation:

$$L_{ext}(q + \delta q, \dot{q} + \delta \dot{q}, \gamma + \delta \gamma, \dot{S} + \delta \dot{S}) \frac{dt'}{dt} = L_{ext}(q, \dot{q}, \gamma, \dot{S}), \quad (4.3.3)$$

which, after a little algebra, can be rewritten as follows:

$$-(1 + \gamma) H(q, \dot{q})(\delta t) + (1 + \gamma) \frac{\partial L}{\partial \dot{q}}(\delta \dot{q}) + (1 + \gamma) \frac{\partial L}{\partial q}(\delta q) - \gamma(\delta \dot{S}) + (L - \dot{S}) \delta \gamma = 0. \quad (4.3.4)$$

If we plug Eq.(4.3.2) into Eq.(4.3.4) we obtain:

$$-(1 + \gamma) \epsilon L(q, \dot{q}) - \gamma(\delta \dot{S}) + (L - \dot{S}) \delta \gamma = 0, \quad (4.3.5)$$

and we have the two following possibilities.

4.3.1 Case $\delta \gamma = 0$

We require $\delta \gamma = 0$ in Eq.(4.3.5) and we are left with:

$$\gamma(\delta \dot{S}) = -(1 + \gamma) \epsilon L(q, \dot{q}). \quad (4.3.6)$$

The most general solution of the previous equation is given by the following expression:

$$\delta S = -\frac{\epsilon(\gamma + 1)}{\gamma} \int_0^t d\tau L(q(\tau), \dot{q}(\tau)). \quad (4.3.7)$$

This in turn implies that the complete transformation is (remember Eq.(4.3.2)):

$$\begin{cases} \delta t = \epsilon \int_0^t d\tau \frac{L(q(\tau), \dot{q}(\tau))}{H(q(\tau), \dot{q}(\tau))} \\ \delta q = 0 \\ \delta S = -\frac{\epsilon(\gamma + 1)}{\gamma} \int_0^t d\tau L(q(\tau), \dot{q}(\tau)) \\ \delta \gamma = 0. \end{cases} \quad (4.3.8)$$

The next step is to find out what is the current associated to the symmetry transformation (4.3.8). It is easy to see that this current is:

$$\begin{aligned} Q_1 &= \frac{\partial L_{ext}}{\partial \dot{q}} \delta q + \frac{\partial L_{ext}}{\partial \dot{S}} \delta S + L_{ext} \delta t \\ &= (1 + \gamma) \delta t \left(-\frac{\partial L}{\partial \dot{q}} \dot{q} + L \right) - \gamma \delta S \\ &= \epsilon (1 + \gamma) \left(-H \int_0^t d\tau \frac{L}{H} + \int_0^t d\tau L \right), \end{aligned} \quad (4.3.9)$$

where in the last step we used the results in Eq.(4.3.8). Now, Noether's theorem states that the current Q_1 is conserved on the shell of the equations of motion. Anyway, it is not difficult to check that Q_1 vanishes on the shell of the equations of motion (H is a constant on this shell and can be passed inside the integral), in such a way that Noether's theorem is satisfied in a trivial way. More interesting is the following possibility.

4.3.2 Case $\delta\gamma \neq 0$

Differently from the previous case we make the choice $\delta\gamma = \epsilon(1 + \gamma)$ and from Eq.(4.3.5) we obtain:

$$\gamma(\delta \dot{S}) + \epsilon \dot{S} (1 + \gamma) = 0; \quad (4.3.10)$$

this in turn implies that:

$$\delta S = - \int_0^t d\tau \frac{\epsilon(1 + \gamma)}{\gamma} \dot{S} + const. \quad (4.3.11)$$

After an integration by parts we can rewrite the previous formula as:

$$\delta S = -\frac{\epsilon(1 + \gamma)}{\gamma} S(t) + \epsilon \int_0^t d\tau \left[\frac{d}{d\tau} \left(\frac{(1 + \gamma)}{\gamma} \right) \right] S(\tau) + const; \quad (4.3.12)$$

We can now summarize the complete transformation in the following way:

$$\begin{cases} \delta t = \epsilon \int_0^t d\tau \frac{L(q(\tau), \dot{q}(\tau))}{H(q(\tau), \dot{q}(\tau))}; \\ \delta q = 0; \\ \delta S = -\frac{\epsilon(1+\gamma)}{\gamma} S(t) + \epsilon \int_0^t d\tau \left[\frac{d}{d\tau} \left(\frac{(1+\gamma)}{\gamma} \right) \right] S(\tau) + const; \\ \delta \gamma = \epsilon(1+\gamma). \end{cases} \quad (4.3.13)$$

The conserved current associated to (4.3.13) is easily found:

$$\begin{aligned} Q_2 &= \frac{\partial L_{ext}}{\partial \dot{q}} \bar{\delta} q + \frac{\partial L_{ext}}{\partial \dot{S}} \bar{\delta} S + L_{ext} \delta t \\ &= \epsilon(1+\gamma) \left(S - H \int_0^t d\tau \frac{L}{H} \right) - \epsilon \gamma \int_0^t d\tau \dot{S} \frac{d}{d\tau} \left(\frac{1+\gamma}{\gamma} \right) + const; \end{aligned} \quad (4.3.14)$$

This current, on the shell of the equations of motion, takes the form:

$$Q_2 = \epsilon(1+\gamma) \left(S - \int_0^t d\tau L \right) + const; \quad (4.3.15)$$

and Noether’s theorem becomes:

$$\frac{dQ_2}{dt} = 0 \implies \frac{dS}{dt} = L. \quad (4.3.16)$$

Let us stop here for a while and open a brief parenthesis. Everybody knows that the Hamilton-Jacobi equation of Classical Mechanics is:

$$\frac{\partial S(q, t)}{\partial t} + H \left(q, \frac{\partial S(q, t)}{\partial q} \right) = 0, \quad (4.3.17)$$

where $S(q, t)$ is called the *Hamilton generating function*. It is easy to check that the previous equation is equivalent to the following set of equations:

$$\begin{cases} \frac{dS(q, t)}{dt} = L(q, \dot{q}), \\ \frac{\partial S(q, t)}{\partial q} = \frac{\partial L(q, \dot{q})}{\partial \dot{q}}; \end{cases} \quad (4.3.18)$$

in fact, the first equation in (4.3.18) can be rewritten as

$$\frac{\partial S(q, t)}{\partial t} + \frac{\partial S(q, t)}{\partial q} \dot{q} - L(q, \dot{q}) = 0, \quad (4.3.19)$$

and if we substitute the second equation of (4.3.18) into (4.3.19), we finally obtain the Hamilton-Jacobi equation (4.3.17). Now we see that the Noether theorem stated in Eq.(4.3.16) corresponds to the first relation in (4.3.18), and therefore we are close to our goal which was to find out the conserved current associated to the MSA symmetry. Unfortunately it is not so simple to complete the project and obtain also the second relation (the missing one in Eq.(4.3.16)) of (4.3.18). The reason is basically twofold. Firstly it is not so clear how to implement on $S(t)$, in a consistent way, a dependence on q at the level of Eq.(4.3.16). The second motivation is conceptually deeper and therefore it is described with full details in the next section.

4.4 MSA as an anholonomic transformation

In the previous sections we have already claimed many times that the MSA-transformation is not a standard symmetry of Classical Systems for several reasons. One of them (maybe the most important) is that this transformation is path-dependent. In particular it transforms the differentials dq in such a way that they cannot be integrated to a diffeomorphism, as we shall show in this section. These kinds of transformations are known in the literature as “anholonomic transformations” [25] and it is useful to spend few lines in describing what are their most important features.

The most general definition of symmetry is a transformation which leaves invariant the equations of motion of a dynamical system. In most cases, this requirement is equivalent to leaving invariant the action of the system at hand, but there are cases in which this is no longer true (the anholonomic transformations are one example). This problem never arises when we deal with diffeomorphisms: for instance consider a dynamical system with Lagrangian $L(q, \dot{q}, t)$ and suppose to make the following transformation:

$$q \longrightarrow q'(q, t); \quad (4.4.1)$$

the transformed Lagrangian is then given by:

$$L'(q', \dot{q}'; t) = L(q, \dot{q}; t). \quad (4.4.2)$$

What we are going to show is that if a function $\tilde{q}(t)$ is a solution of Lagrange equations for L , then the function

$$\tilde{q}'(t) = q'(\tilde{q}, t) \quad (4.4.3)$$

is a solution of the Lagrange equations associated to L' . In fact the transformed

Lagrange derivative¹⁹ is:

$$\left[\frac{d}{dt} \left(\frac{\partial L'(q', \dot{q}'; t)}{\partial \dot{q}'} \right) - \frac{\partial L'(q', \dot{q}'; t)}{\partial q'} \right]_{\bar{q}'(t)}, \quad (4.4.4)$$

and if we use Eq.(4.4.2) in Eq.(4.4.4), we can rewrite it in the following way:

$$\left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \frac{\partial \dot{q}}{\partial \dot{q}'} + \frac{\partial L}{\partial \dot{q}} \frac{d}{dt} \left(\frac{\partial \dot{q}}{\partial \dot{q}'} \right) - \frac{\partial L}{\partial q} \frac{\partial q}{\partial q'} - \frac{\partial L}{\partial \dot{q}} \frac{\partial \dot{q}}{\partial q'} \right]_{\bar{q}(t)}. \quad (4.4.5)$$

Now, inverting Eq.(4.4.1), it is not difficult to show that:

$$\frac{\partial \dot{q}}{\partial \dot{q}'} = \frac{\partial q}{\partial q'}, \quad (4.4.6)$$

and if we substitute Eq.(4.4.6) into Eq.(4.4.5) we obtain:

$$\begin{aligned} \left[\frac{d}{dt} \left(\frac{\partial L'}{\partial \dot{q}'} \right) - \frac{\partial L'}{\partial q'} \right]_{\bar{q}'(t)} = \\ \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} \right]_{\bar{q}(t)} \frac{\partial q}{\partial q'} + \frac{\partial L}{\partial \dot{q}} \left[\frac{d}{dt} \left(\frac{\partial \dot{q}}{\partial q'} \right) - \frac{\partial \dot{q}}{\partial q'} \right]_{\bar{q}(t)}. \end{aligned} \quad (4.4.7)$$

But now, by hypothesis, we know that for the unprimed system the Lagrange equations hold:

$$\left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} \right]_{\bar{q}(t)} = 0, \quad (4.4.8)$$

and therefore the RHS of Eq.(4.4.7) reduces to the second term which is zero because:

$$\frac{d}{dt} \left(\frac{\partial q}{\partial q'} \right) - \frac{\partial \dot{q}}{\partial q'} = \frac{\partial^2 q}{\partial q'^2} \dot{q}' + \frac{\partial^2 q}{\partial t \partial q'} - \frac{\partial}{\partial q'} \left[\frac{\partial q}{\partial q'} \dot{q}' + \frac{\partial q}{\partial t} \right] = 0, \quad (4.4.9)$$

where in the last step we used the fact that $\frac{\partial \dot{q}'}{\partial q'} = 0$. Finally we can write:

$$\left[\frac{d}{dt} \left(\frac{\partial L'}{\partial \dot{q}'} \right) - \frac{\partial L'}{\partial q'} \right]_{\bar{q}'(t)} = 0 \quad (4.4.10)$$

¹⁹Given a function $f(q, \dot{q})$, the Lagrange derivative of $f(q, \dot{q})$ is defined as

$$\frac{d}{dt} \left(\frac{\partial f(q, \dot{q})}{\partial \dot{q}} \right) - \frac{\partial f(q, \dot{q})}{\partial q}.$$

which is precisely what we wanted to prove. Note that we could repeat all the steps above (longer calculations are involved) if we considered, instead of Eq.(4.4.1), a more general transformation like the following:

$$\begin{cases} q' = q'(q, t); \\ t' = t'(t); \end{cases} \quad (4.4.11)$$

but this is not necessary for our purposes and we shall not do it here. This for the diffeomorphisms. On the other hand, a general anholonomic transformation $x^\mu \rightarrow y^\rho$ is defined as:

$$\begin{cases} dx^\mu = e^\mu_\rho(y) dy^\rho \\ \frac{\partial e^\mu_\rho(y)}{\partial y^\sigma} \neq \frac{\partial e^\mu_\sigma(y)}{\partial y^\rho} \end{cases} \quad (4.4.12)$$

where the second condition corresponds to the failure of the Schwarz integrability condition. We can rewrite the first of Eqs.(4.4.12) by introducing a new parameter τ :

$$x^\mu(\tau) - x^\mu(0) = \int_0^\tau d\eta e^\mu_\rho(y) \frac{dy^\rho}{d\eta}. \quad (4.4.13)$$

If the Schwarz integrability condition were satisfied, Eq.(4.4.13) could be integrated (at least in principle²⁰) to obtain $x(\sigma) = x(q(\sigma))$ and we would get a diffeomorphism; in general this is not possible. This means that an anholonomic transformation can relate a system which obeys the Lagrange equations to a system where these equations are no longer satisfied. This introduces us to the main point of this section. Up to now we have always called the MSA-transformation a symmetry (even if particular) because its effect is the universal rescaling of the classical action. Now we can ask: is this sufficient to call it a symmetry, or is there anything else which must be taken into account? Actually we shall show that the MSA-transformation is anholonomic and the rescaling of the action, in this case, does not imply the invariance of the equations of motion. To show that, we could proceed in two ways. One possibility is to rephrase the MSA transformation in a formalism in which the time t is at the same level of q (Maupertuis formalism) and prove the failure of the Schwarz integrability condition (4.4.12). The other possibility is to prove directly that the MSA transformation does not leave invariant the equations of motion of the system at hand. In the sequel we shall follow this strategy.

First of all, let us recast the MSA transformation in a form similar to (4.4.12).

²⁰The fact that it can be integrated does not mean that the result has necessarily an analytic expression.

Remember that we have:

$$\begin{cases} \delta t = \epsilon \int_0^t d\tau \frac{L(q, \dot{q})}{H(q, \dot{q})} \\ \delta q = 0; \end{cases} \quad (4.4.14)$$

and from the first equation we get:

$$t'(t) - t = \epsilon \int_0^t d\tau \frac{L(q, \dot{q})}{H(q, \dot{q})} \quad (4.4.15)$$

which implies:

$$dt' = dt + \epsilon dt R(q, \dot{q}) \quad (4.4.16)$$

where we have denoted by $R(q, \dot{q})$ the following ratio:

$$R(q, \dot{q}) = \frac{L(q, \dot{q})}{H(q, \dot{q})} = \frac{m\dot{q}^2 - 2V(q)}{m\dot{q}^2 + 2V(q)}. \quad (4.4.17)$$

From Eq.(4.4.16) we also obtain:

$$\frac{dt'}{dt} = 1 + \epsilon R(q, \dot{q}), \quad (4.4.18)$$

which allows us to write:

$$\dot{q}' = \frac{dq'(t')}{dt'} = \frac{dq'(t')}{dt} \frac{dt}{dt'} = \frac{dq(t)}{dt} \frac{dt}{dt'} = \dot{q} [1 - \epsilon R(q, \dot{q})], \quad (4.4.19)$$

which in turn implies:

$$\ddot{q}' = \frac{d}{dt'} \{ \dot{q} [1 - \epsilon R(q, \dot{q})] \} = \ddot{q} [1 - 2\epsilon R(q, \dot{q})] - \epsilon \dot{q} \left(\frac{\partial R}{\partial q} \dot{q} + \frac{\partial R}{\partial \dot{q}} \ddot{q} \right). \quad (4.4.20)$$

Now we have all the ingredients to check whether the equations of motion of the old (the unprimed) system are really invariant, i.e. if they have the same form as the new (primed) ones. Let us start from these last equations which are simply:

$$m\ddot{q}' = -\frac{\partial V(q')}{\partial q'}, \quad (4.4.21)$$

and substitute Eq.(4.4.20) into Eq.(4.4.21); we get:

$$m\ddot{q}' = m \left\{ \ddot{q} [1 - 2\epsilon R(q, \dot{q})] - \epsilon \dot{q} \left(\frac{\partial R}{\partial q} \dot{q} + \frac{\partial R}{\partial \dot{q}} \ddot{q} \right) \right\} = -\frac{\partial V(q')}{\partial q'} = -\frac{\partial V(q)}{\partial q}. \quad (4.4.22)$$

Next, if we use $\frac{\partial V}{\partial q} = -m\ddot{q}$ in Eq.(4.4.22) we obtain:

$$2\ddot{q} R + \dot{q}^2 \frac{\partial R}{\partial q} + \dot{q}\ddot{q} \frac{\partial R}{\partial \dot{q}} = 0; \quad (4.4.23)$$

and we must check whether this last equation holds on the shell of the old equations of motion. To do that we need the two following expressions:

$$\frac{\partial R}{\partial q} = -\frac{4m\dot{q}^2 \frac{\partial V}{\partial q}}{(m\dot{q}^2 + 2V)^2}; \quad (4.4.24)$$

$$\frac{\partial R}{\partial \dot{q}} = \frac{8m\dot{q} V}{(m\dot{q}^2 + 2V)^2}; \quad (4.4.25)$$

which, when substituted in Eq.(4.4.23), yield:

$$2m^2\ddot{q}\dot{q}^4 - 8\ddot{q} V^2(q) - 4m\dot{q}^4 \frac{\partial V(q)}{\partial q} + 8m\dot{q}^2 \ddot{q} V(q) = 0. \quad (4.4.26)$$

Now, if we use again the unprimed equations of motion $\frac{\partial V}{\partial q} = -m\ddot{q}$ we can simplify Eq.(4.4.26) and reduce it to the following form:

$$3m^2\dot{q}^4 + 4m\dot{q}^2 V(q) - 4V^2(q) = 0; \quad (4.4.27)$$

but this equation is not satisfied for every classical system. Therefore we have proven that in general the equations of motion of a classical system are not invariant under the MSA transformation. This means that in this case the rescaling of the action is not a sufficient condition for leaving invariant the equations of motion (the Lagrange equations), which is the main property of a symmetry transformation. Therefore we can conclude that the \mathcal{Q}_S -transformation which we have introduced at the beginning of this chapter cannot be interpreted as a generalization to the space $\widetilde{\mathcal{M}}$ of a universal symmetry of Classical Mechanics at the level of (q, t) .

5. The Conformal Extension of the CPI

In the previous chapter we have introduced two transformations: the first one (generated by \mathcal{Q}_s) rescales the action of the CPI while the second (the MSA) rescales the standard action of Classical Mechanics. We have seen that in both cases the formalism is not easy to manage because on one hand \mathcal{Q}_s induces a noncanonical transformation on $(\varphi^a, c^a, \lambda_a, \bar{c}_a)$ while, on the other hand, the MSA is even anholonomic and path dependent. In this chapter we want to analyze another rescaling of time which is much easier to handle because of its canonical structure. This transformation is known as *superconformal transformation* and is the result of the composition of a conformal transformation together with a Susy. In the following sections we shall present a model which exhibits a kind of superconformal invariance derived from the classical Susy of the CPI. Our hope is that this particular model may be a playground to tackle the more general problems of the transformations introduced in the previous chapter.

5.1 Conformal Mechanics

In this section we briefly describe the model which was originally presented in [15]. It is a one dimensional model described by a configuration $q(t)$ where t is the base space which in our context will be always understood as the time. The action is:

$$S = \int dt \frac{1}{2} \left[\dot{q}^2 - \frac{g}{q^2} \right], \quad (5.1.1)$$

where g is a dimensional constant. It is not difficult to check that this Lagrangian is invariant under the following transformations:

$$t' = \frac{\alpha t + \beta}{\gamma t + \delta}; \quad (5.1.2)$$

$$q'(t') = \frac{q(t)}{(\gamma t + \delta)}; \quad (5.1.3)$$

$$(\text{with } \alpha\delta - \beta\gamma = 1);$$

which are the conformal transformations in 0+1 dimensions²¹. We have confined in Appendix C.1 a brief review of the Conformal Group and its representations; for our purposes here, it is enough to say that a generic transformation (5.1.2) is a composition of the following three transformations:

$$t' = \alpha^2 t \quad \text{dilations,} \quad (5.1.4)$$

$$t' = t + \beta \quad \text{time-translations,} \quad (5.1.5)$$

$$t' = \frac{t}{\gamma t + 1} \quad \text{special-conformal transformations.} \quad (5.1.6)$$

We can apply Noether's theorem to the following transformations and what we get is that the conserved charges are the following:

$$H = \frac{1}{2} \left(p^2 + \frac{g}{q^2} \right); \quad (5.1.7)$$

$$D = tH - \frac{1}{4}(qp + pq); \quad (5.1.8)$$

$$K = t^2 H - \frac{1}{2}t(qp + pq) + \frac{1}{2}q^2. \quad (5.1.9)$$

Using the quantum commutator $[q, p] = i$, we see that the three charges above obey the following algebra:

$$[H, D] = iH; \quad (5.1.10)$$

$$[K, D] = -iK; \quad (5.1.11)$$

$$[H, K] = 2iD. \quad (5.1.12)$$

The reader should not be surprised by the fact that D and K do not commute with H , even if they are constants of motion. In fact they depend explicitly on time and therefore Noether's theorem — which claims:

$$0 = \frac{d}{dt} D = \frac{\partial}{\partial t} D + i[H, D]; \quad (5.1.13)$$

$$0 = \frac{d}{dt} K = \frac{\partial}{\partial t} K + i[H, K]; \quad (5.1.14)$$

²¹ “0+1” refers to the spatial+temporal dimensions of the base space. In our case we have 0 spatial dimensions and 1 (i.e. the time t) temporal dimension.

does not imply that $[H, D] = 0$ or $[H, K] = 0$. Anyway, since the algebra above does not involve the time t explicitly, the same commutation relations are satisfied also by the same operators evaluated at time $t = 0$, namely:

$$H_0 = \frac{1}{2} \left[p^2 + \frac{g}{q^2} \right], \quad (5.1.15)$$

$$D_0 = -\frac{1}{4} [qp + pq], \quad (5.1.16)$$

$$K_0 = \frac{1}{2} q^2, \quad (5.1.17)$$

which will turn useful in the following. The algebra (5.1.10)-(5.1.12) is that of the group $SO(2, 1)$, which is isomorphic (see Appendix C.1 for the details) to the conformal group in 0+1 dimensions.

5.2 Conformal Mechanics and General relativity

Before going to the supersymmetric extensions of Conformal Mechanics, we give a brief account of the reason why this model has gained again much attention in the context of black holes and General Relativity [12][45].

First of all we consider the metric of an *extreme* Reissner-Nordström black hole: for the reader who is not familiar with it we give a brief account of this topic in Appendix D.1. For our purposes here, it is enough to write down the form of the metric:

$$ds^2 = - \left(1 - \frac{M}{r} \right)^2 dt^2 + \left(1 - \frac{M}{r} \right)^{-2} dr^2 + r^2 d\Omega^2, \quad (5.2.1)$$

where:

$$d\Omega^2 = \sin^2 \theta d\varphi^2 + d\theta^2 \quad (5.2.2)$$

is the usual expression of the solid angle. As a first step we change the coordinates by introducing a new variable $\rho \equiv r - M$: the metric (5.2.1) takes the form:

$$ds^2 = - \left(1 + \frac{M}{\rho} \right)^2 dt^2 + \left(1 + \frac{M}{\rho} \right)^2 [d\rho^2 + \rho^2 d\Omega^2]. \quad (5.2.3)$$

The effect of this change of variable is simple: we have shifted the singularity from $r_o = M$ to $\rho_o = 0$. Therefore, in the near-horizon region ($\rho \cong 0$), we can approximate the previous expression with the following:

$$ds^2 = - \left(\frac{\rho}{M} \right)^2 dt^2 + \left(\frac{M}{\rho} \right)^2 d\rho^2 + M^2 d\Omega^2. \quad (5.2.4)$$

If we finally introduce two further new variables (z, τ) defined as:

$$\begin{cases} \rho =: M e^{-z} \cos \tau \\ t =: M e^z \tan \tau, \end{cases} \quad (5.2.5)$$

after a little algebra we find that (5.2.4) becomes:

$$ds^2 = M^2 [-d\tau^2 + \cos^2 \tau dz^2 + d\Omega^2], \quad (5.2.6)$$

which goes under the name of Bertotti-Robinson metric. The first two terms of the RHS are invariant under $SO(2, 1)$ while $d\Omega^2$ is clearly invariant under $SO(3)$. The metric exhibits an overall $SO(2, 1) \times SO(3)$ invariance, which is commonly called in the literature $AdS_2 \times S_2$ (see Appendix D.3 for the details). Therefore we already have a hint about how the conformal group enters the game, because — as we have seen previously — the conformal mechanics in $0 + 1$ dimensions is invariant under transformations of the group $SO(2, 1)$. These subject has recently obtained a big interest because of the famous Maldacena's conjecture [45], which claims that the large N limit of a conformally invariant theory in d -dimensions (the boundary) is equivalent to supergravity on $d + 1$ -dimensional AdS space (the bulk) times a compact manifold (which in the maximally supersymmetric case is a sphere).

Let us now proceed with the description of the relation between the Conformal Mechanics (5.1.1) and the Reissner-Nordström metric (5.2.4). Our next step is to determine the Hamiltonian which governs the dynamics of a particle in the near-horizon region (i.e. $\rho \cong 0$) in which the approximation (5.2.4) holds. First of all we make the last change of variables by introducing a new parameter R defined as follows:

$$\frac{\rho}{M} =: \left(\frac{2M}{R} \right)^2 \quad (5.2.7)$$

which implies:

$$ds^2 = - \left(\frac{2M}{R} \right)^4 dt^2 + \left(\frac{2M}{R} \right)^2 dR^2 + M^2 d\Omega^2. \quad (5.2.8)$$

From the previous equation it is easy to see that the metric coefficients $g_{\mu\nu}$ do not depend on t and therefore (see Appendix D.2) p_t is a constant of the motion, which is just the Hamiltonian we are looking for. We can calculate it explicitly in the following way (other methods are possible, of course). First of all we exploit the fact that we have a constraint on the momenta:

$$(p_\mu - qA_\mu)g^{\mu\nu}(p_\nu - qA_\nu) + m^2 = 0 \quad (5.2.9)$$

where q and m are the charge and the mass of the particle moving in the near-horizon region, and A_μ is the electromagnetic potential generated by the black

hole. Since the latter is a *static* charge sitting at $r = 0$ we have that the only non vanishing component of A_μ is:

$$A_0(\rho) = -\frac{Q}{r(\rho)} = -\frac{1}{1 + \frac{\rho}{M}}. \quad (5.2.10)$$

In the near-horizon region ($\rho \cong 0$) we can rewrite it as:

$$A_0(\rho) = -\left(1 - \frac{\rho}{M}\right) \cong \left(\frac{2M}{R}\right)^2, \quad (5.2.11)$$

where in the last step we have used the fact that an electromagnetic potential does not feel (at least at the classical level) additive constants. Now, if we plug (5.2.11) into (5.2.9) and we solve for p_t , after a little algebra we find:

$$H \equiv -p_t = \left(\frac{2M}{R}\right)^2 \left\{ \left[m^2 + \frac{1}{4M^2} (R^2 p_R^2 + 4L^2) \right]^{\frac{1}{2}} - q \right\}, \quad (5.2.12)$$

where

$$L^2 \equiv p_\theta^2 + \frac{p_\varphi^2}{\sin^2 \theta} \quad (5.2.13)$$

is clearly the angular momentum which is another constant of motion both at the classical and the quantum level (where becomes $l(l+1)\hbar^2$). Finally we can make more manifest the conformal symmetry if we rewrite the Hamiltonian (5.2.12) as:

$$H = \frac{p_R^2}{2f} + \frac{g}{2fR^2}, \quad (5.2.14)$$

where f and g are respectively:

$$f \equiv \frac{1}{2} \left[\left(m^2 + \frac{R^2 p_R^2 + 4L^2}{4M^2} \right)^{\frac{1}{2}} + q \right] \quad (5.2.15)$$

$$g = 4M^2(m^2 - q^2) + 4L^2. \quad (5.2.16)$$

Then, apart from the factor f , we can recognize the Hamiltonian (5.1.7) of the Conformal Mechanics.

5.3 Superconformal Mechanics

In this section we briefly review the first supersymmetric extension of Conformal Mechanics, which was provided in Ref.[26]. This extension of model (5.1.1) is tailored on the celebrated Supersymmetric Quantum Mechanics invented by Witten [59] and subsequently developed by others [14][49]. The Hamiltonian is:

$$H_{Susy} = \frac{1}{2} \left(p^2 + \frac{g}{q^2} + \frac{\sqrt{g}}{q^2} [\psi^\dagger, \psi]_- \right) \quad (5.3.1)$$

where ψ, ψ^\dagger are Grassmannian variables whose anticommutator is $[\psi, \psi^\dagger]_+ = 1$. As one can notice, in H_{Susy} there is a bosonic piece which is the conformal Hamiltonian (5.1.7), plus a Grassmannian part. Note that the equations of motion of “ q ” have an extra piece with respect to the equations of motion of the old Conformal Mechanics [15].

To make contact with Supersymmetric Quantum Mechanics let us notice that H_{Susy} can be written as:

$$H_{Susy} = \frac{1}{2} \left(p^2 + \left(\frac{dW}{dq} \right)^2 - [\psi^\dagger, \psi]_- \frac{d^2W}{dq^2} \right) \quad (5.3.2)$$

(which is the Hamiltonian of Susy-QM) where the superpotential W turns out to be:

$$W(q) = \sqrt{g} \log q. \quad (5.3.3)$$

The two Supersymmetry charges associated to the Hamiltonian (5.3.1) are easily found:

$$Q = \psi^\dagger \left(-ip + \frac{dW}{dq} \right), \quad (5.3.4)$$

$$Q^\dagger = \psi \left(ip + \frac{dW}{dq} \right), \quad (5.3.5)$$

whose commutator closes on the Hamiltonian:

$$[Q, Q^\dagger]_+ = 2H_{Susy}. \quad (5.3.6)$$

It is interesting to see what we obtain when we combine a supersymmetric transformation with a conformal one generated by the (H, K, D) elements of the $SO(2, 1)$ algebra (5.1.10)-(5.1.12). We get what is called a *superconformal* transformation. In order to understand this better let us list the following eight operators:

TABLE 1

H	$= \frac{1}{2} \left[p^2 + \frac{g + 2\sqrt{g}B}{q^2} \right];$
D	$= -\frac{[q, p]_+}{4};$
K	$= \frac{q^2}{2};$
B	$= \frac{[\psi^\dagger, \psi]_-}{2};$
Q	$= \psi^\dagger \left[-ip + \frac{\sqrt{g}}{q} \right];$
Q^\dagger	$= \psi \left[ip + \frac{\sqrt{g}}{q} \right];$
S	$= \psi^\dagger q;$
S^\dagger	$= \psi q.$

The algebra of these operators is closed and given in **TABLE 2**:

TABLE 2

$[H, D] = iH;$	$[K, D] = -iK;$	$[H, K] = 2iD;$
$[Q, H] = 0;$	$[Q^\dagger, H] = 0;$	$[Q, D] = \frac{i}{2}Q;$
$[Q^\dagger, K] = S^\dagger;$	$[Q, K] = -S;$	$[Q^\dagger, D] = \frac{i}{2}Q^\dagger;$
$[S, K] = 0;$	$[S^\dagger, K] = 0;$	$[S, D] = -\frac{i}{2}S;$
$[S^\dagger, D] = -\frac{i}{2}S^\dagger;$	$[S, H] = -Q;$	$[S^\dagger, H] = Q^\dagger;$
$[Q, Q^\dagger] = 2H;$	$[S, S^\dagger] = 2K;$	
$[B, S] = S;$	$[B, S^\dagger] = -S^\dagger;$	
$[Q, S^\dagger] = \sqrt{g} - B + 2iD;$	$[B, Q] = Q;$	$[B, Q^\dagger] = -Q^\dagger;$

all other commutators are zero or derivable from these by Hermitian conjugation. The square-brackets $[(...), (...)]$ in the algebra above are *graded*-commutators and from now on we shall omit the subindex + or – as we did before. They are commutators or anticommutators according to the Grassmannian nature of the operators entering the brackets.

As is well known, a *superconformal* transformation is a combination of a supersymmetry transformation and a conformal one. We see from the algebra above that the commutators of the supersymmetry generators (Q, Q^\dagger) with the three conformal generators (H, K, D) generate two new operators (S, S^\dagger) . Including these operators we generate an algebra which is closed provided that we introduce the operator B of **TABLE 1**. This is the last operator we need.

5.4 A New Supersymmetric Extension of Conformal Mechanics

In the previous Section we have described the supersymmetric extension of Conformal Mechanics which is based on the Susy-QM model of Witten. In this Section we want to obtain a new supersymmetric extension of the action (5.1.1) by use of another strategy. In fact, as we have seen in Section 1.3, the Hamiltonian $\tilde{\mathcal{H}}$ of the Classical Path Integral is automatically supersymmetric, regardless of the particular shape of $H(\varphi)$ from which it is built. Therefore, if we choose as $H(\varphi)$ the conformal Hamiltonian of Eq.(5.1.7) we will obtain a model of Superconformal

Mechanics which is basically different from that tailored on the Susy-QM model. Thus, as a first step, we insert the H of Eq.(5.1.7) into the $\tilde{\mathcal{H}}$ of Eq.(1.1.8). The result is:

$$\tilde{\mathcal{H}} = \lambda_q p + \lambda_p \frac{g}{q^3} + i\bar{c}_q c^p - 3i\bar{c}_p c^q \frac{g}{q^4} \quad (5.4.1)$$

where the indices $(\dots)^q$ and $(\dots)^p$ on the variables (λ, c, \bar{c}) replace the indices $(\dots)^a$ which appeared in the general formalism. In fact, as the system is one-dimensional, the index “ $(\dots)^a$ ” can only indicate the variables “ (p, q) ” and that is why we use (p, q) as index. The two supersymmetric charges of Eqs.(1.3.8) and (1.3.9) are in this case

$$Q_H = Q_{BRS} + \beta \left(\frac{g}{q^3} c^q - p c^p \right) \quad (5.4.2)$$

$$\overline{Q}_H = \overline{Q}_{BRS} + \beta \left(\frac{g}{q^3} \bar{c}_p + p c_q \right). \quad (5.4.3)$$

It was one of the central points of the original paper [15] on Conformal Mechanics that the Hamiltonians of the system could be, beside H_0 of Eq.(5.1.15), also D_0 or K_0 of Eqs.(5.1.16)(5.1.17) or any linear combination of them. In the same manner as we built the Lie-derivative $\tilde{\mathcal{H}}$ associated to H_0 , we can also build the Lie-derivatives associated to the flow generated by D_0 and K_0 . We just have to insert²² D_0 or K_0 in place of H as superpotential in the $\tilde{\mathcal{H}}$ of Eq.(1.1.8). If we denote the associated Lie-derivatives by $\tilde{\mathcal{D}}_0$ and $\tilde{\mathcal{K}}_0$, what we get is:

$$\tilde{\mathcal{D}}_0 = \frac{1}{2} [\lambda_p p - \lambda_q q + i(\bar{c}_p c^p - \bar{c}_q c^q)] \quad (5.4.4)$$

$$\tilde{\mathcal{K}}_0 = -\lambda_p q - i\bar{c}_p c^q \quad (5.4.5)$$

The construction is best illustrated in Figure 1.

As it is easy to prove, both $\tilde{\mathcal{D}}_0$ and $\tilde{\mathcal{K}}_0$ are supersymmetric. It is possible in fact to introduce the following charges:

$$Q_D = Q_{BRS} + \gamma(qc^p + pc^q); \quad (5.4.6)$$

$$\overline{Q}_D = \overline{Q}_{BRS} + \gamma(p\bar{c}_p - q\bar{c}_q); \quad (5.4.7)$$

$$Q_K = Q_{BRS} - \alpha qc^q; \quad (5.4.8)$$

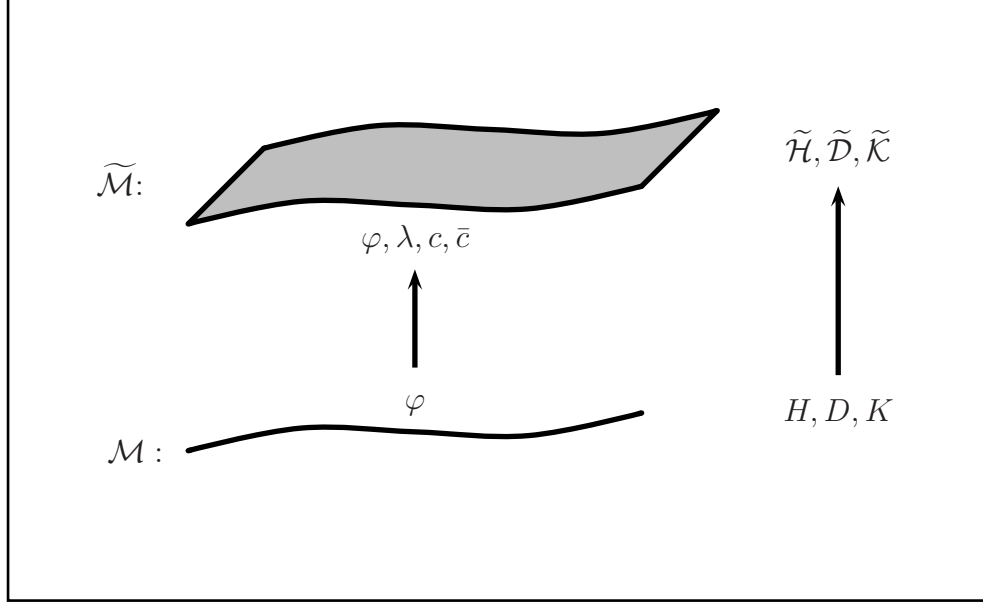
$$\overline{Q}_K = \overline{Q}_{BRS} - \alpha q\bar{c}_p; \quad (5.4.9)$$

(γ and α play the same role as β for H) which close on $\tilde{\mathcal{D}}_0$ and $\tilde{\mathcal{K}}_0$:

$$[Q_D, \overline{Q}_D] = 4i\gamma\tilde{\mathcal{D}}_0 \quad (5.4.10)$$

$$[Q_K, \overline{Q}_K] = 2i\alpha\tilde{\mathcal{K}}_0. \quad (5.4.11)$$

²²We will neglect ordering problem in the expression of D_0 because we are doing a classical theory. The sub-index “ $(\dots)_0$ ” that we will put on $\tilde{\mathcal{D}}$ and $\tilde{\mathcal{K}}$ below is to indicate that they were built from D_0 and K_0 .

Figure 5.1: The correspondence between \mathcal{M} and $Super\mathcal{M}$.

One further point to notice is that the conformal algebra of Eqs.(5.1.10)-(5.1.12) is now realized, via the commutators (1.2.3) of our formalism, by the $(\widetilde{\mathcal{H}}, \widetilde{\mathcal{D}}_0, \widetilde{\mathcal{K}}_0)$ and not by the old functions (H, D_0, K_0) . In fact, via these new commutators, we get:

$$[\widetilde{\mathcal{H}}, \widetilde{\mathcal{D}}_0] = i\widetilde{\mathcal{H}}; \quad [\widetilde{\mathcal{K}}_0, \widetilde{\mathcal{D}}_0] = -i\widetilde{\mathcal{K}}_0; \quad [\widetilde{\mathcal{H}}, \widetilde{\mathcal{K}}_0] = 2i\widetilde{\mathcal{D}}_0; \quad (5.4.12)$$

$$[H, D_0] = 0; \quad [K_0, D_0] = 0; \quad [H, K_0] = 0. \quad (5.4.13)$$

The next thing to find out, assuming $\widetilde{\mathcal{H}}$ as basic Hamiltonian and Q_H and \overline{Q}_H as supersymmetry charges, is to perform the commutators between supersymmetries and conformal operators so to get the superconformal generators. It is easy to work this out and we get:

$$[Q_H, \widetilde{\mathcal{D}}_0] = i(Q_H - Q_{BRS}); \quad [\overline{Q}_H, \widetilde{\mathcal{D}}_0] = i(\overline{Q}_H - \overline{Q}_{BRS}); \quad (5.4.14)$$

$$[Q_H, \widetilde{\mathcal{K}}_0] = \frac{i\beta}{\gamma}(Q_D - Q_{BRS}); \quad [\overline{Q}_H, \widetilde{\mathcal{K}}_0] = \frac{i\beta}{\gamma}(\overline{Q}_D - \overline{Q}_{BRS}). \quad (5.4.15)$$

From what we have above we realize immediately the role of the Q_D and \overline{Q}_D : besides being the “square roots” of $\widetilde{\mathcal{D}}_0$ they are also (combined with the Q_{BRS} and \overline{Q}_{BRS}) the generators of the superconformal transformations. It is also a simple calculation to evaluate the commutators between the various “supercharges” $Q_H, \overline{Q}_D, \overline{Q}_K, \overline{Q}_H, Q_D, Q_K$:

$$[Q_H, \bar{Q}_D] = i\beta\tilde{\mathcal{H}} + 2i\gamma\tilde{\mathcal{D}}_0 - 2\beta\gamma H; \quad [\bar{Q}_H, Q_D] = i\beta\tilde{\mathcal{H}} + 2i\gamma\tilde{\mathcal{D}}_0 + 2\beta\gamma H; \quad (5.4.16)$$

$$[Q_K, \bar{Q}_D] = i\alpha\tilde{\mathcal{K}}_0 + 2i\gamma\tilde{\mathcal{D}}_0 + 2\alpha\gamma K_0; \quad [\bar{Q}_K, Q_D] = i\alpha\tilde{\mathcal{K}}_0 + 2i\gamma\tilde{\mathcal{D}}_0 - 2\alpha\gamma K_0; \quad (5.4.17)$$

$$[Q_H, \bar{Q}_K] = i\beta\tilde{\mathcal{H}} + i\alpha\tilde{\mathcal{K}}_0 - 2\alpha\beta D_0; \quad [\bar{Q}_H, Q_K] = i\beta\tilde{\mathcal{H}} + i\alpha\tilde{\mathcal{K}}_0 + 2\alpha\beta D_0. \quad (5.4.18)$$

From the RHS of these expressions we see that one needs also the old functions (H, D_0, K_0) in order to close the algebra.

The complete set of operators which close the algebra is listed in the following table:

TABLE 3

$\begin{aligned} \tilde{\mathcal{H}} &= \lambda_q p + \lambda_p \frac{g}{q^3} + i\bar{c}_q c^p - 3i\bar{c}_p c^q \frac{g}{q^4}; \\ \tilde{\mathcal{K}}_0 &= -\lambda_p q - i\bar{c}_p c^q; \\ \tilde{\mathcal{D}}_0 &= \frac{1}{2}[\lambda_p p - \lambda_q q + i(\bar{c}_p c^p - \bar{c}_q c^q)]; \\ Q_{BRS} &= i(\lambda_q c^q + \lambda_p c^p); \\ Q_H &= Q_{BRS} + \beta \left(\frac{g}{q^3} c^q - p c^p \right); \\ Q_K &= Q_{BRS} - \alpha q c^q; \\ Q_D &= Q_{BRS} + \gamma(q c^p + p c^q); \end{aligned}$	$\begin{aligned} H &= \frac{1}{2} \left(p^2 + \frac{g}{q^2} \right); \\ K_0 &= \frac{1}{2} q^2; \\ D_0 &= -\frac{1}{2} q p; \\ \bar{Q}_{BRS} &= i(\lambda_p \bar{c}_q - \lambda_q \bar{c}_p); \\ \bar{Q}_H &= \bar{Q}_{BRS} + \beta \left(\frac{g}{q^3} \bar{c}_p + p \bar{c}_q \right); \\ \bar{Q}_K &= \bar{Q}_{BRS} - \alpha q \bar{c}_p; \\ \bar{Q}_D &= \bar{Q}_{BRS} + \gamma(p \bar{c}_p - q \bar{c}_q). \end{aligned}$
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The complete algebra among these generators is given in **TABLE 4**, where the missing commutators are zero and the $Q_{(\dots)}$ appearing in the table can be any of the following operators: Q_{BRS}, Q_H, Q_D, Q_K (the same holds for $\bar{Q}_{(\dots)}$). Obviously all commutators are understood between quantities calculated at the same time.

We notice that for our supersymmetric extension we need 14 charges (see **TABLE 3**) to close the algebra, while in the extension of Ref. [26] one needs only 8 charges (see **TABLE 1**). This is so not only because ours is an $N = 2$ supersymmetry (while the one of [26] is an $N = 1$) but also because of the totally different character of the model.

TABLE 4

$[\tilde{\mathcal{H}}, \tilde{\mathcal{D}}_0] = i\tilde{\mathcal{H}};$	$[\tilde{\mathcal{K}}, \tilde{\mathcal{D}}_0] = -i\tilde{\mathcal{K}}_0;$	$[\tilde{\mathcal{H}}, \tilde{\mathcal{K}}_0] = 2i\tilde{\mathcal{D}}_0;$
$[Q_H, \tilde{\mathcal{H}}] = 0;$	$[\bar{Q}_H, \tilde{\mathcal{H}}] = 0;$	$[Q_H, \bar{Q}_H] = 2i\beta\tilde{\mathcal{H}};$
$[Q_H, \tilde{\mathcal{D}}_0] = i(Q_H - Q_{BRS});$	$[\bar{Q}_H, \tilde{\mathcal{D}}_0] = i(\bar{Q}_H - \bar{Q}_{BRS});$	
$[Q_H, \tilde{\mathcal{K}}_0] = i\beta\gamma^{-1}(Q_D - Q_{BRS});$	$[\bar{Q}_H, \tilde{\mathcal{K}}_0] = i\beta\gamma^{-1}(\bar{Q}_D - \bar{Q}_{BRS});$	
$[Q_{BRS}, \tilde{\mathcal{H}}] = [\bar{Q}_{BRS}, \tilde{\mathcal{H}}] = 0;$	$[Q_{BRS}, \tilde{\mathcal{K}}_0] = [\bar{Q}_{BRS}, \tilde{\mathcal{K}}_0] = 0;$	$[Q_{BRS}, \tilde{\mathcal{D}}_0] = [\bar{Q}_{BRS}, \tilde{\mathcal{D}}_0] = 0;$
$[Q_D, \tilde{\mathcal{H}}] = -2i\gamma\beta^{-1}(Q_H - Q_{BRS});$	$[\bar{Q}_D, \tilde{\mathcal{H}}] = -2i\gamma\beta^{-1}(\bar{Q}_H - \bar{Q}_{BRS});$	
$[Q_D, \tilde{\mathcal{K}}_0] = 2i\gamma\alpha^{-1}(Q_K - Q_{BRS});$	$[\bar{Q}_D, \tilde{\mathcal{K}}_0] = 2i\gamma\alpha^{-1}(\bar{Q}_K - \bar{Q}_{BRS});$	
$[Q_D, \tilde{\mathcal{D}}_0] = 0;$	$[\bar{Q}_D, \tilde{\mathcal{D}}_0] = 0;$	$[Q_D, \bar{Q}_D] = 4i\gamma\tilde{\mathcal{D}}_0;$
$[Q_K, \tilde{\mathcal{H}}] = -i\alpha\gamma^{-1}(Q_D - Q_{BRS});$	$[\bar{Q}_K, \tilde{\mathcal{H}}] = -i\alpha\gamma^{-1}(\bar{Q}_D - \bar{Q}_{BRS});$	
$[Q_K, \tilde{\mathcal{D}}_0] = -i(Q_K - Q_{BRS});$	$[\bar{Q}_K, \tilde{\mathcal{D}}_0] = -i(\bar{Q}_K - \bar{Q}_{BRS});$	
$[Q_K, \tilde{\mathcal{K}}_0] = 0;$	$[\bar{Q}_K, \tilde{\mathcal{K}}_0] = 0;$	$[Q_K, \bar{Q}_K] = 2i\alpha\tilde{\mathcal{K}}_0;$
$[Q_H, \bar{Q}_D] = i\beta\tilde{\mathcal{H}} + 2i\gamma\tilde{\mathcal{D}}_0 - 2\beta\gamma H;$	$[\bar{Q}_H, Q_D] = i\beta\tilde{\mathcal{H}} + 2i\gamma\tilde{\mathcal{D}}_0 + 2\beta\gamma H;$	
$[Q_K, \bar{Q}_D] = i\alpha\tilde{\mathcal{K}}_0 + 2i\gamma\tilde{\mathcal{D}}_0 + 2\alpha\gamma K;$	$[\bar{Q}_K, Q_D] = i\alpha\tilde{\mathcal{K}}_0 + 2i\gamma\tilde{\mathcal{D}}_0 - 2\alpha\gamma K;$	
$[Q_H, \bar{Q}_K] = i\beta\tilde{\mathcal{H}} + i\alpha\tilde{\mathcal{K}}_0 - 2\alpha\beta D;$	$[\bar{Q}_H, Q_K] = i\beta\tilde{\mathcal{H}} + i\alpha\tilde{\mathcal{K}}_0 + 2\alpha\beta D;$	
$[Q_H, \bar{Q}_{BRS}] = [\bar{Q}_H, Q_{BRS}] = i\beta\tilde{\mathcal{H}};$	$[Q_K, \bar{Q}_{BRS}] = [\bar{Q}_K, Q_{BRS}] = i\alpha\tilde{\mathcal{K}}_0;$	
$[Q_D, \bar{Q}_{BRS}] = [\bar{Q}_D, Q_{BRS}] = 2i\gamma\tilde{\mathcal{D}}_0;$		
$[Q_{(\dots)}, H] = \beta^{-1}(Q_{BRS} - Q_H);$	$[\bar{Q}_{(\dots)}, H] = \beta^{-1}(\bar{Q}_{BRS} - \bar{Q}_H);$	
$[Q_{(\dots)}, D_0] = (2\gamma)^{-1}(Q_{BRS} - Q_D);$	$[\bar{Q}_{(\dots)}, D_0] = (2\gamma)^{-1}(\bar{Q}_{BRS} - \bar{Q}_D);$	
$[Q_{(\dots)}, K_0] = \alpha^{-1}(Q_{BRS} - Q_K);$	$[\bar{Q}_{(\dots)}, K_0] = \alpha^{-1}(\bar{Q}_{BRS} - \bar{Q}_K);$	
$[\tilde{\mathcal{H}}, H] = 0;$	$[\tilde{\mathcal{K}}_0, K_0] = 0;$	$[\tilde{\mathcal{D}}_0, D_0] = 0;$
$[\tilde{\mathcal{H}}, K_0] = [H, \tilde{\mathcal{K}}_0] = 2iD_0;$	$[\tilde{\mathcal{H}}, D_0] = [H, \tilde{\mathcal{D}}_0] = iH;$	$[\tilde{\mathcal{D}}_0, K_0] = [D_0, \tilde{\mathcal{K}}_0] = iK_0.$

5.5 Study of the two Superconformal Algebras

A Lie superalgebra [40] is an algebra made of even E_n and odd O_α generators whose graded commutators look like:

$$[E_m, E_n] = F_{mn}^p E_p; \quad (5.5.1)$$

$$[E_m, O_\alpha] = G_{m\alpha}^\beta O_\beta; \quad (5.5.2)$$

$$[O_\alpha, O_\beta] = C_{\alpha\beta}^m E_m; \quad (5.5.3)$$

and where the structure constants $F_{mn}^p, G_{m,\alpha}^\beta, C_{\alpha,\beta}^m$ satisfy generalized Jacobi identities.

One can interpret the relation (5.5.2) by saying that the even part of the algebra has a representation on the odd part. This is clear if we consider the odd part as a vector space and that the even part acts on this vector space via the graded

commutators. The structure constants $F_{m\alpha}^\beta$ are then the matrix elements which characterize the representations.

For superconformal algebras the usual folklore says that the even part of the algebra has his conformal subalgebra represented spinorially on the odd part. The reasoning roughly goes as follows: the odd part of the algebra must contain the supersymmetry generators which transform as spinors under the Lorentz group which is a subgroup of the conformal algebra. So it is impossible that the whole conformal algebra is represented non-spinorially on the odd part.

Actually this line of reasoning is true in a relativistic context in which the supersymmetry is a true relativistic supersymmetry and the charges must carry a spinor index due to their nature. Instead, in our non-relativistic point particle case, the charges do not carry any space-time index and so we do not have as a consequence that necessarily the even part of the algebra is represented spinorially on the odd part. It can happen but it can also not happen. In this respect we will analyze the superalgebras of the two supersymmetric extensions of conformal mechanics seen here, the one of [26] and ours.

Let us start from the one of Ref. [26] which is given in **TABLE 1**. The conformal subalgebra \mathcal{G}_0 of the even part can be organized in an $SO(2, 1)$ form as follows:

$$\mathcal{G}_0 : \begin{cases} B_1 = \frac{1}{2} \left[\frac{K}{a} - aH \right] \\ B_2 = D \\ J_3 = \frac{1}{2} \left[\frac{K}{a} + aH \right] \end{cases}$$

where a is the same parameter introduced in [15] with a physical dimension of time.

The odd part \mathcal{G}_1 is:

$$\mathcal{G}_1 : \begin{cases} Q \\ Q^\dagger \\ S \\ S^\dagger \end{cases}$$

It is easy to work out, using the results of **TABLE 2**, the action of \mathcal{G}_0 on \mathcal{G}_1 . The result is summarized in the following table:

TABLE 5

$[B_1, Q] = \frac{1}{2a}S;$	$[B_2, Q] = -\frac{i}{2}Q;$	$[J_3, Q] = \frac{1}{2a}S;$
$[B_1, Q^\dagger] = -\frac{1}{2a}S^\dagger;$	$[B_2, Q^\dagger] = -\frac{i}{2}Q^\dagger;$	$[J_3, Q^\dagger] = -\frac{1}{2a}S^\dagger;$
$[B_1, S] = -\frac{a}{2}Q;$	$[B_2, S] = \frac{i}{2}S;$	$[J_3, S] = \frac{a}{2}Q;$
$[B_1, S^\dagger] = \frac{a}{2}Q^\dagger;$	$[B_2, S^\dagger] = \frac{i}{2}S^\dagger;$	$[J_3, S^\dagger] = -\frac{a}{2}Q^\dagger.$

As we said before, in order to act with the even part of the algebra on the odd part, we have to consider the odd part as a vector space. Let us then introduce the following “vectors”:

$$|q\rangle \equiv Q + Q^\dagger \quad (5.5.4)$$

$$|p\rangle \equiv S - S^\dagger \quad (5.5.5)$$

$$|r\rangle \equiv Q - Q^\dagger \quad (5.5.6)$$

$$|s\rangle \equiv S + S^\dagger; \quad (5.5.7)$$

they label a 4-dimensional vector space. On these vectors we act via the commutators, for example:

$$B_1|q\rangle \equiv [B_1, Q + Q^\dagger] \quad (5.5.8)$$

It is then immediate to realize from **TABLE 5** that the 2-dim. space with basis $(|q\rangle, |p\rangle)$ forms a closed space under the action of the even part of the algebra so it carries a 2-dim. representation and the same holds for the space $(|r\rangle, |s\rangle)$. We can immediately check which kind of representation this is: Let us take the Casimir operator of the algebra \mathcal{G}_0 which is $\mathcal{C} = B_1^2 + B_2^2 - J_3^2$ and apply it to a state of one of the two 2-dim. representations:

$$\begin{aligned} \mathcal{C}|q\rangle &= [B_1, [B_1, Q + Q^\dagger]] + [B_2, [B_2, Q + Q^\dagger]] - [J_3, [J_3, Q + Q^\dagger]] \\ &= -\frac{3}{4}(Q + Q^\dagger) \\ &= -\frac{3}{4}|q\rangle. \end{aligned} \quad (5.5.9)$$

This factor $\frac{3}{4} = -\frac{1}{2}(\frac{1}{2} + 1)$ indicates that the $(|q\rangle, |p\rangle)$ space carries a spinorial representation. It is possible to prove the same for the other space.

Let us now turn the same crank for our supersymmetric extension of conformal mechanics. Looking at the **TABLE 3** of our operators, we can organize the even part \mathcal{G}_0 , as follows:

TABLE 6 (\mathcal{G}_0)

$B_1 = \frac{1}{2} \left(\frac{\tilde{\mathcal{K}}}{a} - a\tilde{\mathcal{H}} \right);$	$P_1 = 2D;$
$B_2 = \tilde{\mathcal{D}};$	$P_2 = aH - \frac{K}{a};$
$J_3 = \frac{1}{2} \left(\frac{\tilde{\mathcal{K}}}{a} + a\tilde{\mathcal{H}} \right);$	$P_0 = aH + \frac{K}{a}.$

The LHS is the usual $SO(2,1)$ while the RHS is formed by three translations because they commute among themselves. So the overall algebra is the Euclidean group $E(2,1)$.

The odd part of our superalgebra is made of 8 operators (see **TABLE 3**) which are:

TABLE 7 (\mathcal{G}_1)

$Q_H;$	$\overline{Q}_H;$
$Q_K;$	$\overline{Q}_K;$
$Q_D;$	$\overline{Q}_D;$
$Q_{BRS};$	$\overline{Q}_{BRS}.$

As we did before in **TABLE 5** for the model of [26], we will now evaluate for our model the action of \mathcal{G}_0 on \mathcal{G}_1 . The result is summarized in the next table, where

for simplicity we have made the choice $a = \sqrt{\frac{\beta}{\alpha}}$ and $\eta \equiv \frac{\gamma}{\sqrt{\alpha\beta}}$.

TABLE 8

$[B_1, Q_H] = \frac{i}{2\eta}(Q_{BRS} - Q_D);$	$[B_1, \overline{Q}_H] = \frac{i}{2\eta}(\overline{Q}_{BRS} - \overline{Q}_D);$
$[B_1, Q_K] = \frac{i}{2\eta}(Q_{BRS} - Q_D);$	$[B_1, \overline{Q}_K] = \frac{i}{2\eta}(\overline{Q}_{BRS} - \overline{Q}_D);$
$[B_1, Q_D] = -i\eta(Q_H + Q_K - 2Q_{BRS});$	$[B_1, \overline{Q}_D] = -i\eta(\overline{Q}_H + \overline{Q}_K - 2\overline{Q}_{BRS});$
$[B_1, Q_{BRS}] = 0;$	$[B_1, \overline{Q}_{BRS}] = 0;$
$[B_2, Q_H] = i(Q_{BRS} - Q_H);$	$[B_2, \overline{Q}_H] = i(\overline{Q}_{BRS} - \overline{Q}_H);$
$[B_2, Q_K] = i(Q_K - Q_{BRS});$	$[B_2, \overline{Q}_K] = i(\overline{Q}_K - \overline{Q}_{BRS});$
$[B_2, Q_D] = 0;$	$[B_2, \overline{Q}_D] = 0;$
$[B_2, Q_{BRS}] = 0;$	$[B_2, \overline{Q}_{BRS}] = 0;$
$[J_3, Q_H] = \frac{i}{2\eta}(Q_{BRS} - Q_D);$	$[J_3, \overline{Q}_H] = \frac{i}{2\eta}(\overline{Q}_{BRS} - \overline{Q}_D);$
$[J_3, Q_K] = -\frac{i}{2\eta}(Q_{BRS} - Q_D);$	$[J_3, \overline{Q}_K] = -\frac{i}{2\eta}(\overline{Q}_{BRS} - \overline{Q}_D);$
$[J_3, Q_D] = i\eta(Q_H - Q_K);$	$[J_3, \overline{Q}_H] = i\eta(\overline{Q}_H - \overline{Q}_K);$
$[J_3, Q_{BRS}] = 0;$	$[J_3, \overline{Q}_{BRS}] = 0;$
$[P_1, Q_{(\dots)}] = \gamma^{-1}(Q_D - Q_{BRS});$	$[P_1, \overline{Q}_{(\dots)}] = -\gamma^{-1}(\overline{Q}_D - \overline{Q}_{BRS});$
$[P_2, Q_{(\dots)}] = \gamma^{-1}\eta(Q_H - Q_K);$	$[P_2, \overline{Q}_{(\dots)}] = -\gamma^{-1}\eta(\overline{Q}_H - \overline{Q}_K);$
$[P_0, Q_{(\dots)}] = \gamma^{-1}\eta(Q_H + Q_K - 2Q_{BRS});$	$[P_0, \overline{Q}_{(\dots)}] = -\gamma^{-1}\eta(\overline{Q}_H + \overline{Q}_K - 2\overline{Q}_{BRS}).$

As we have to represent the conformal subalgebra of \mathcal{G}_0 (see **TABLE 6**) on the vector space \mathcal{G}_1 of **TABLE 7** it is easy to realize from **TABLE 8** that the following three vectors

$$\begin{cases} |q_H\rangle = (Q_H - Q_{BRS}) - (\overline{Q}_H - \overline{Q}_{BRS}) \\ |q_K\rangle = (Q_K - Q_{BRS}) - (\overline{Q}_K - \overline{Q}_{BRS}) \\ |q_D\rangle = \eta^{-1}[(Q_D - Q_{BRS}) - (\overline{Q}_D - \overline{Q}_{BRS})] \end{cases} \quad (5.5.10)$$

make an irreducible representation of the conformal subalgebra. In fact, using **TABLE 8**, we get:

$$\begin{cases} B_1|q_H\rangle = -\frac{i}{2}|q_D\rangle \\ B_2|q_H\rangle = -i|q_H\rangle \\ J_3|q_H\rangle = -\frac{i}{2}|q_D\rangle \\ B_1|q_K\rangle = -\frac{i}{2}|q_D\rangle \\ B_2|q_K\rangle = i|q_K\rangle \\ J_3|q_K\rangle = \frac{i}{2}|q_D\rangle \\ B_1|q_D\rangle = -i(|q_H\rangle + |q_K\rangle) \\ B_2|q_D\rangle = 0 \\ J_3|q_D\rangle = i(|q_H\rangle - |q_K\rangle). \end{cases} \quad (5.5.11)$$

Having three vectors in this representation we presume it is an ‘integer’ spin representation, but to be sure let us apply the Casimir operator on a vector. The Casimir is given, as before, by: $\mathcal{C} = B_1^2 + B_2^2 - J_3^2$ but we must remember to use as B_1 , B_2 and J_3 the operators contained in **TABLE 6**. It is then easy to check that

$$\mathcal{C}|q_H\rangle = -2|q_H\rangle. \quad (5.5.12)$$

The same we get for the other two vectors $|q_K\rangle, |q_D\rangle$, so the eigenvalue in the equation above is $-2 = -1(1 + 1)$ and this indicates that those vectors make a “spin 1” representation.

In the same way as before it is easy to prove that the following three vectors:

$$\begin{cases} |\widetilde{q_H}\rangle = (Q_H - Q_{BRS}) + (\overline{Q}_H - \overline{Q}_{BRS}) \\ |\widetilde{q_K}\rangle = (Q_K - Q_{BRS}) + (\overline{Q}_K - \overline{Q}_{BRS}) \\ |\widetilde{q_D}\rangle = (Q_D - Q_{BRS}) + (\overline{Q}_D - \overline{Q}_{BRS}) \end{cases} \quad (5.5.13)$$

make another irreducible representation with “spin 1”.

Of course, as the vector space \mathcal{G}_1 of **TABLE 7** is 8-dimensional and up to now we have used only 6 vectors to build the two integer representations, we expect that there must be some other representations which can be built using the two remaining vectors. In fact it is so. We can build the following two other vectors:

$$|q_{BRS}\rangle = Q_{BRS} - \overline{Q}_{BRS} \quad (5.5.14)$$

$$|\widetilde{q_{BRS}}\rangle = Q_{BRS} + \overline{Q}_{BRS} \quad (5.5.15)$$

and it is easy to see that each of them carries a representation of spin zero:

$$\mathcal{C}|q_{BRS}\rangle = \mathcal{C}|\widetilde{q_{BRS}}\rangle = 0. \quad (5.5.16)$$

So we can conclude that our vector space \mathcal{G}_1 carries a reducible representation of the conformal algebra made of two spin one and two spin zero representations.

We wanted to do this analysis in order to underline a further difference between our supersymmetric extension and the one of [26] whose odd part \mathcal{G}_1 , as we showed before, carries two “spin $\frac{1}{2}$ ” representations.

5.6 Superspace Formulation of the Model

In this Section we use the techniques introduced in Section (1.4) and we give a superspace formulation of our new Superconformal Mechanics. Proceeding in the same way as we did for the Q_H and \overline{Q}_H charges, it is a long but easy procedure to give a superspace representation also for the charges $Q_D, \overline{Q}_D, Q_K, \overline{Q}_K$ of

Eqs.(5.4.6)-(5.4.9). This long derivation is contained in Appendix E.1 and the result is:

$$\mathcal{Q}_K = -\frac{\partial}{\partial\theta} - \alpha \omega^{ad} K_{db} \bar{\theta} \quad (5.6.1)$$

$$\bar{\mathcal{Q}}_K = \frac{\partial}{\partial\bar{\theta}} + \alpha \omega^{ad} K_{db} \theta \quad (5.6.2)$$

$$\mathcal{Q}_D = -\frac{\partial}{\partial\theta} - 2\gamma \omega^{ad} D_{db} \bar{\theta} \quad (5.6.3)$$

$$\bar{\mathcal{Q}}_D = \frac{\partial}{\partial\bar{\theta}} + 2\gamma \omega^{ad} D_{db} \theta \quad (5.6.4)$$

where the matrices K_{db} and D_{db} are:

$$K_{db} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}; \quad D_{db} = -\frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (5.6.5)$$

(the repeated indices in Eqs.(5.6.1)-(5.6.4) are summed). The matrices K_{db} and D_{db} are 2×2 just because the symplectic matrix itself ω^{ab} is 2×2 in our case. The conformal mechanics system in fact has just a pair of phase-space variables (p, q) and the index in the φ^a -phase space variables can take only 2 values to indicate either q or p (see the Eqs. of motion (1.1.1)).

From the expressions of $\mathcal{Q}_K, \bar{\mathcal{Q}}_K, \mathcal{Q}_D, \bar{\mathcal{Q}}_D$ above, we see that they have two free indices. This implies that those operators “turn” the various superfields in the sense that they turn a Φ^q into combinations of Φ^q and Φ^p and viceversa. This is something the other charges did not do.

Reached this point, we should stop and think a little bit about this superspace representation. We gave the superspace representation of the various charges $(Q_D, \bar{Q}_D, Q_K, \bar{Q}_K)$ of Eqs.(5.4.6)-(5.4.9) which were linked to the $\tilde{\mathcal{D}}_0, \tilde{\mathcal{K}}_0$ of Eqs.(5.4.4)(5.4.5). But these last quantities were built using the D_0 and K_0 that is the D and K at $t = 0$. If we had used, in building the $\tilde{\mathcal{D}}_0, \tilde{\mathcal{K}}_0$, the \tilde{D} and \tilde{K} at $t \neq 0$ of Eqs.(5.1.8) and (5.1.9), we would have obtained a $\tilde{\mathcal{D}}$ and a $\tilde{\mathcal{K}}$ different from those of Eqs.(5.4.4)(5.4.5) and which would have had an explicit dependence on t . Consequently also the associated supersymmetric charges $(Q_D^t, \bar{Q}_D^t, Q_K^t, \bar{Q}_K^t)$, having extra terms depending on t , would be different from those of Eqs.(5.4.6)-(5.4.9). Being these charges different, also their superspace representations will be different from those given in Eqs.(5.6.1)-(5.6.4). The difference at the level of superspace is crucial because it involves t which is part of superspace.

Let us then start this over-all process by first building the explicitly t -dependent $\tilde{\mathcal{D}}$ and $\tilde{\mathcal{K}}$ from the following operators D and K :

$$H = H_0 \quad (5.6.6)$$

$$D = tH + D_0 \quad (5.6.7)$$

$$K = t^2H + 2tD_0 + K_0 \quad (5.6.8)$$

from which we get:

$$\tilde{\mathcal{D}} = t\tilde{\mathcal{H}} + \tilde{\mathcal{D}}_0 \quad (5.6.9)$$

$$\tilde{\mathcal{K}} = t^2\tilde{\mathcal{H}} + 2t\tilde{\mathcal{D}}_0 + \tilde{\mathcal{K}}_0. \quad (5.6.10)$$

It is easy to understand why these relations hold by remembering the manner we got the Lie-derivatives out of the superpotentials. The explicit form of $\tilde{\mathcal{D}}$ in terms of $(\varphi^a, c^a, \lambda_a, \bar{c}_a)$ can be obtained from (5.6.9) once we insert the $\tilde{\mathcal{H}}$ and $\tilde{\mathcal{D}}_0$ whose explicit form we already had in Eqs.(5.4.1) and (5.4.4). The same for $\tilde{\mathcal{K}}$. Let us now turn to the form of the associated fermionic charges which we will indicate as $(Q_D^t, Q_K^t, \bar{Q}_D^t, \bar{Q}_K^t)$ where the index “ $(...)^t$ ” is to indicate their explicit dependence on t . As it is shown in formula (E.1.1) of Appendix E.1, the Q_D and Q_K can be written using the charges N_D and N_K of (E.1.2) and the Q_{BRS} . As it is only the $N_{(...)}$ and not the Q_{BRS} which pull in quantities like D, K which may depend explicitly on time, we should only concentrate on the $N_{(...)}$. From their definition (see Eq.(E.1.2)):

$$N_D = c^a \partial_a D; \quad N_K = c^a \partial_a K \quad (5.6.11)$$

we see that applying the operator $c^a \partial_a$ on both sides of Eqs.(5.6.7)(5.6.8), we get:

$$N_D^t = tN_H + N_D \quad (5.6.12)$$

$$N_K^t = t^2 N_H + 2tN_D + N_K. \quad (5.6.13)$$

The next step is to write the Q_D^t and Q_K^t . As they are given in formula (E.1.1), using that equation and (5.6.12)(5.6.13) above we get:

$$Q_D^t = Q_{BRS} - 2\gamma N_D^t \quad (5.6.14)$$

$$Q_K^t = Q_{BRS} - \alpha N_K^t. \quad (5.6.15)$$

In a similar manner, via Eq.(E.1.9) and applying the operator $\bar{c}_a \omega^{ab} \partial_b$ to Eqs.(5.6.7) and (5.6.8), we get \bar{Q}_D^t and \bar{Q}_K^t :

$$\bar{Q}_D^t = \bar{Q}_{BRS} + 2\gamma \bar{N}_D^t \quad (5.6.16)$$

$$\bar{Q}_K^t = \bar{Q}_{BRS} + \alpha \bar{N}_K^t. \quad (5.6.17)$$

We shall not write down explicitly the expressions of $(Q_D^t, Q_K^t, \bar{Q}_D^t, \bar{Q}_K^t)$ in terms of $(\varphi^a, c^a, \lambda_a, \bar{c}_a)$ because we have already in Eqs.(5.4.6)-(5.4.9) and (E.1.2)(E.1.9)

the expressions²³ of the various charges $(Q_D, Q_K, \bar{Q}_D, \bar{Q}_K, N_D, N_K, \bar{N}_D, \bar{N}_K)$ which make up, according to Eqs.(5.6.14)-(5.6.17), the new time dependent charges. The next step is to obtain the superspace version of $(Q_D^t, Q_K^t, \bar{Q}_D^t, \bar{Q}_K^t)$. Following a procedure identical to the one explained in detailed in the Appendix for the time-independent charges it is easy to get them and, via their anticommutators, to derive the superspace version of $\tilde{\mathcal{D}}$ and $\tilde{\mathcal{K}}$. All these operators are listed in the table below:

TABLE 9

$\tilde{\mathcal{H}}$	$= i \frac{\partial}{\partial t};$
$\tilde{\mathcal{D}}$	$= it \frac{\partial}{\partial t} - \frac{i}{2} \sigma_3;$
$\tilde{\mathcal{K}}$	$= it^2 \frac{\partial}{\partial t} - it \sigma_3 - i \sigma_-;$
\mathcal{Q}_D^t	$= -\frac{\partial}{\partial \theta} - 2\gamma \bar{\theta} t \frac{\partial}{\partial t} + \gamma \bar{\theta} \sigma_3;$
$\bar{\mathcal{Q}}_D^t$	$= \frac{\partial}{\partial \bar{\theta}} + 2\gamma \theta t \frac{\partial}{\partial t} - \gamma \theta \sigma_3;$
\mathcal{Q}_K^t	$= -\frac{\partial}{\partial \theta} - \alpha \bar{\theta} t^2 \frac{\partial}{\partial t} + \alpha t \bar{\theta} \sigma_3 + \alpha \bar{\theta} \sigma_-;$
$\bar{\mathcal{Q}}_K^t$	$= \frac{\partial}{\partial \bar{\theta}} + \alpha \theta t^2 \frac{\partial}{\partial t} - \alpha t \theta \sigma_3 - \alpha \theta \sigma_-;$
\mathcal{H}	$= \bar{\theta} \theta \frac{\partial}{\partial t};$
\mathcal{D}	$= \bar{\theta} \theta (t \frac{\partial}{\partial t} - \frac{1}{2} \sigma_3);$
\mathcal{K}	$= \bar{\theta} \theta (t^2 \frac{\partial}{\partial t} - t \sigma_3 - \sigma_-).$

In the previous table σ_3 and σ_- are the Pauli matrices:

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (5.6.18)$$

The reasons for the presence of these two-dimensional matrices has already been explained in the paragraph below Eq.(5.6.5).

The last three operators listed in **TABLE 9** are the superspace version of the old (H, D, K) . To get this representation we used again and again the rules given by Eq.(1.4.2). As their representation looks quite unusual, we have reported the details of their derivations in Appendix E.2.

²³Only be careful in using the D_0 and K_0 in Eqs.(E.1.1) and (E.1.9).

5.7 Exact Solution of the Supersymmetric Model

The original conformal mechanical model was solved exactly in Eq.(2.35) of Ref.[15]. The solution is given by the relation:

$$q^2(t) = 2t^2H - 4tD_0 + 2K_0. \quad (5.7.1)$$

As (H, D_0, K_0) are constants of motion, once their values are assigned we stick them in Eq.(5.7.1), and we get a relation between “ q ” (on the LHS of (5.7.1)) and “ t ” on the RHS. This is the solution of the equation of motion with “initial conditions” given by the values we assign to the constants of motion (H, D_0, K_0) . The reader may object that we should give only two constant values (corresponding to the initial conditions $(q(0), \dot{q}(0))$) and not three. Actually the three values assigned to (H, D_0, K_0) are not arbitrary because, as it was proven in Eq.(2-36) of Ref. [15], these three quantities are linked by the constraint:

$$(HK_0 - D_0^2) = \frac{g}{4} \quad (5.7.2)$$

where “ g ” is the coupling which entered the original Hamiltonian (see Eq.(5.1.7) of the present paper). Having one constraint among the three constants of motion brings them down to two.

The proof of the relation (5.7.1) above is quite simple. On the RHS, as the (H, D_0, K_0) are constants of motion, we can replace them with their time dependent expression (H, D, K) (see Eqs.(5.6.6)-(5.6.8)), which are explicitly:

$$H = \frac{1}{2} \left(\dot{q}^2(t) + \frac{g}{q^2(t)} \right) \quad (5.7.3)$$

$$D = tH - \frac{1}{2}q(t)\dot{q}(t) \quad (5.7.4)$$

$$K = t^2H - t q(t)\dot{q}(t) + \frac{1}{2}q^2 \quad (5.7.5)$$

Inserting these expressions in the RHS of Eq.(5.7.1) we get immediately the LHS. From Eqs.(5.7.3)-(5.7.5) it is also easy to see the relations between the initial conditions $(q(0), \dot{q}(0))$ and the constants (H, D_0, K_0) ; in fact we have:

$$H = H(t=0) = \frac{1}{2} \left(\dot{q}^2(0) + \frac{g}{q^2(0)} \right) \quad (5.7.6)$$

$$D = D_0 = -\frac{1}{2} (q(0)\dot{q}(0)) \quad (5.7.7)$$

$$K = K_0 = \frac{1}{2}q^2(0). \quad (5.7.8)$$

From the relations above we see that, inverting them, we can express $(q(0), \dot{q}(0))$ in term of (H, D_0, K_0) . The constraint (5.7.2) is already involved in the expression of (H, D_0, K_0) in terms of $(q(0), \dot{q}(0))$.

What we want to do in this section is to see whether a relation analogous to (5.7.3) exists also for our supersymmetric extension or in general whether the supersymmetric system can be solved exactly. The answer is *yes* and it is based on a very simple trick.

Let us first remember Eq.(1.4.15) which told us how $\tilde{\mathcal{H}}$ and H are related:

$$i \int H(\Phi) d\theta d\bar{\theta} = \tilde{\mathcal{H}}. \quad (5.7.9)$$

The same relation holds for $\tilde{\mathcal{D}}_0$ and $\tilde{\mathcal{K}}_0$ with respect to D_0 and K_0 as it is clear from the explanation given in the paragraph above Eqs.(5.4.4)(5.4.5) that:

$$i \int D_0(\Phi) d\theta d\bar{\theta} = \tilde{\mathcal{D}}_0 \quad (5.7.10)$$

$$i \int K_0(\Phi) d\theta d\bar{\theta} = \tilde{\mathcal{K}}_0. \quad (5.7.11)$$

Of course the same kind of relations holds for the explicitly time-dependent quantities of Eqs.(5.6.7)-(5.6.10):

$$i \int D(\Phi) d\theta d\bar{\theta} = \tilde{\mathcal{D}} \quad (5.7.12)$$

$$i \int K(\Phi) d\theta d\bar{\theta} = \tilde{\mathcal{K}}. \quad (5.7.13)$$

Let us now build the following quantity:

$$2t^2 H(\Phi) - 4t D(\Phi) + 2K(\Phi). \quad (5.7.14)$$

This is functionally the RHS of Eq.(5.7.1) with the superfield Φ^a replacing the normal phase-space variable φ^a . It is then clear that the following relation holds:

$$(\Phi^a)^2 = 2t^2 H(\Phi) - 4t D(\Phi) + 2K(\Phi). \quad (5.7.15)$$

The reason it holds is because, in the proof of the analogous one in q -space (Eq.(5.7.1)), the only thing we used was the functional form of the (H, D, K) that was given by Eqs.(5.7.3)-(5.7.5). So that relation holds irrespective of the arguments, φ or Φ , which enter our functions provided that the functional form of them remains the same.

Let us first remember the form of Φ^q which appears²⁴ on the LHS of Eq.(5.7.15):

$$\Phi^q(t, \theta, \bar{\theta}) = q(t) + \theta c^q(t) + \bar{\theta} \omega^{qp} \bar{c}_p(t) + i\bar{\theta}\theta \omega^{qp} \lambda_p(t). \quad (5.7.16)$$

Let us now expand in θ and $\bar{\theta}$ the LHS and RHS of Eq.(5.7.15) and compare the terms with the same power of θ and $\bar{\theta}$.

The RHS is:

$$(\Phi^q)^2 = q^2(t) + \theta[2q(t)c^q(t)] + \bar{\theta}[2q(t)\bar{c}_p(t)] + \bar{\theta}\theta[2iq(t)\lambda_p(t) + 2c^q(t)\bar{c}_p(t)]. \quad (5.7.17)$$

The LHS is instead:

$$2t^2 H(\Phi) - 4tD(\Phi) + 2K(\Phi) = 2t^2 H(\varphi) - 4tD(\varphi) + 2K(\varphi) + \theta[2t^2 N_H^t + 4tN_D^t + 2N_K^t] - \bar{\theta}[2t^2 \bar{N}_H^t - 4t\bar{N}_D^t + 2\bar{N}_K^t] + i\bar{\theta}\theta[2t^2 \tilde{\mathcal{H}} - 4t\tilde{\mathcal{D}} + 2\tilde{\mathcal{K}}], \quad (5.7.18)$$

where the $N_H, \bar{N}_H, N_D^t, N_K^t$ are defined in Eqs.(5.6.12)(5.6.13), while the \bar{N}_D^t, \bar{N}_K^t are the time-dependent versions²⁵ of the operators defined in Eqs.(E.1.10) and (E.1.11). It is a simple exercise to show that all these functions ($H, D, K, N_{(\dots)}, \bar{N}_{(\dots)}, \tilde{\mathcal{H}}, \tilde{\mathcal{K}}, \tilde{\mathcal{D}}$) are constants of motion in the enlarged space $(\varphi^a, c^a, \lambda_a, \bar{c}_a)$.

If we now compare the RHS of Eq.(5.7.17) and Eq.(5.7.18) and equate terms with the same power of θ and $\bar{\theta}$, we get (by writing the N and \bar{N} explicitly):

$$q^2(t) = 2t^2 H(\varphi) - 4tD(\varphi) + 2K(\varphi); \quad (5.7.19)$$

$$2q(t)c^q(t) = \left[2t^2 \frac{\partial H}{\partial \varphi^a} - 4t \frac{\partial D}{\partial \varphi^a} + 2 \frac{\partial K}{\partial \varphi^a} \right] c^a; \quad (5.7.20)$$

$$2q(t)\bar{c}_p(t) = \left[2t^2 \frac{\partial H}{\partial \varphi^a} - 4t \frac{\partial D}{\partial \varphi^a} + 2 \frac{\partial K}{\partial \varphi^a} \right] \omega^{ab} \bar{c}_b; \quad (5.7.21)$$

$$i2q(t)\lambda_p(t) + 2c^q(t)\bar{c}_p(t) = -i \left[2t^2 \tilde{\mathcal{H}} - 4t\tilde{\mathcal{D}} + 2\tilde{\mathcal{K}} \right]. \quad (5.7.22)$$

We notice immediately that Eq.(5.7.19) is the same as the one of the original paper [15] and solves the motion for “ q ”. Given this solution we plug it in Eq.(5.7.20) and, since on the RHS we have the N -functions which are constants, once these constants are assigned we get the motion of c^a . Next we assign three constant values to the \bar{N} -functions which appear on the RHS of Eq.(5.7.21), then we plug

²⁴The index $(\dots)^q$ is not a substitute for the index “ a ” but it indicates, as we said many times before, the first half of the indices “ a ”. Let us remember in fact that the first half of the φ^a are just the configurational variables q^i which in our case of a 1-dim. system is just one variable.

²⁵By “time-dependent versions” we mean that they are related to the time independent ones in the same manner as the N^t -functions were via Eqs.(5.6.12)(5.6.15).

in the solution for q given by Eq.(5.7.19) and so we get the trajectory for \bar{c}_p . Finally we do the same in Eq.(5.7.21) and get the trajectory of λ_p .

The solution for the momentum-quantities $(p, c^p, \bar{c}_q, \lambda_q)$ can be obtained via their definition in terms of the previous variables.

The reader may be puzzled by the fact that in the space $(\varphi^a, c^a, \lambda_a, \bar{c}_a)$ we have 8 variables but we have to give 12 constants of motion: $(H, D_0, K_0, N_{(\dots)}, \bar{N}_{(\dots)}, \tilde{\mathcal{H}}, \tilde{\mathcal{K}}, \tilde{\mathcal{D}})$ to get the solutions from Eqs.(5.7.19)-(5.7.22). The point is that, like in the case of the standard conformal mechanics [15], we have constraints among the constants of motion. We have already one constraint and it is given by Eq.(5.7.2). The others can be obtained in the following manner: let us apply the operator $c^a \partial_a$ on both sides of Eq.(5.7.2) and what we get is the following relation:

$$N_H K_0 + N_K H - 2N_D D_0 = 0 \quad (5.7.23)$$

which is a constraint for the N -functions. Let us now do the same applying on both sides of Eq.(5.7.2) the operator $\bar{c}_a \omega^{ab} \partial_b$. What we get is:

$$\bar{N}_H K_0 + \bar{N}_K H - 2\bar{N}_D D_0 = 0 \quad (5.7.24)$$

which is a constraint among the \bar{N} -functions. Finally let us apply Q_{BRS} to Eq.(5.7.24) and we get:

$$i\tilde{\mathcal{H}}K_0 + i\tilde{\mathcal{K}}H - 2i\tilde{\mathcal{D}}D_0 - \bar{N}_H N_K - \bar{N}_K N_H + 2\bar{N}_D N_D = 0 \quad (5.7.25)$$

which is a constraint among the $\tilde{\mathcal{H}}, \tilde{\mathcal{D}}, \tilde{\mathcal{K}}$.

So we have 4 constraints (5.7.25)(5.7.24)(5.7.23)(5.7.2) which bring down the constants of motion to be specified in $(\varphi^a, c^a, \lambda_a, \bar{c}_a)$ from 12 to 8 as we expected.

Conclusions

In this thesis we have continued the analysis of the formalism of the Classical Path Integral introduced in the literature few years ago. The CPI-Hamiltonian exhibits many symmetries, some of which have already been understood geometrically. In this thesis we have continued this project and we have focused on some other symmetries which seem to have some interesting implications in several issues like — for instance — the study of the geometry of the classical trajectories, the analysis of the ergodic character of a classical system, the quantization mechanism etc.

The first symmetry which we have analyzed is the classical $N = 2$ supersymmetry. This symmetry is *universal*, in the sense that it does not depend on the particular form of the Lagrangian of the system at hand. To understand its geometrical meaning we have built an extension of the CPI-Lagrangian, which exhibits the supersymmetry as a *local* symmetry. This model is therefore a *gauge* theory and consequently only a subset of the overall Hilbert space turns out to be *physical*. In particular, we have discovered that this physical Hilbert space is in one-to-one correspondence with the *equivariant forms* derived from the action of the 1-parameter Hamiltonian evolution group on the phase space. Our goal was to exploit this correspondence to study the geometry of the space of the classical trajectories. In fact there is an important formula which relates the equivariant cohomology $H_G(\mathcal{M})$ built from the action of a group G on a manifold \mathcal{M} to the de-Rham cohomology of the quotient space $H(\mathcal{M}/G)$. The latter is the main instrument to analyze the geometry of $H(\mathcal{M}/G)$ which in our case (G is the Hamiltonian evolution group) is the space of the classical trajectories. However there is still some work to be done in this direction because the correspondence above is valid only if the action of the group G on \mathcal{M} is free, which means that there are no fixed points. This is not always true in our case but we hope to find other useful tools for studying the space of classical orbits.

Besides the relation to the space of classical trajectories, there is also another point where the classical susy can play an important role. In fact, in the literature, there are some attempts to relate the classical supersymmetry to the ergodicity character of a classical system. What was proved in Ref.[35] was that if susy is unbroken the system is ergodic (*susy unbroken* \implies *ergodicity*), but the authors could

not prove the reverse, basically because their analysis should have been restricted to the constant-energy surfaces, where the ergodic characterization makes sense. Here we have tried to make a step ahead along this direction and we have modified the CPI-Lagrangian inserting by hand the fixed-energy constraint. What we have noticed is that the $N = 2$ supersymmetry reduces to an $N = 1$ and hopefully it is this supersymmetry which should be used as an instrument to test the ergodic phase of the system at hand. However this is still an open problem.

There is another symmetry which we have analyzed in detail and is strictly tied to the classical supersymmetry. In fact in the superspace $(t, \theta, \bar{\theta})$ its generators D_H and \bar{D}_H are represented by the covariant derivatives associated to the classical susy charges Q_H and \bar{Q}_H . Here we have shown that the transformation generated by D_H and \bar{D}_H is very similar to the well known κ -symmetry introduced in the literature 20 years ago in the context of the relativistic superparticles. Following the same lines as for the susy, we have built a local extension of the CPI-Lagrangian which has D_H and \bar{D}_H as Noether's charges associated to the gauge symmetry. We have noticed that in our nonrelativistic case there is no problem (differently from the relativistic case) in separating 1st-class from 2nd-class constraints. Moreover the physical Hilbert space associated to this gauge theory is either the space of the Gibbs states (describing the canonical ensemble) or the space of the distributions built of constants of motion only, according to the form of the coupling we choose in the gauge Lagrangian.

There is also a further universal symmetry of the CPI formalism which we have discovered in this work and which — we think — can be crucial in the understanding of the quantization process. Differently from the previous symmetries we have mentioned so far, this one is not canonical and the analytic form of the generator can be found only in the superspace $(t, \theta, \bar{\theta})$. Nevertheless this symmetry (we have called it \mathcal{Q}_s) is very interesting because its effect is a rescaling of the overall CPI-action, and this seems to play a crucial role in the quantization process. In fact the rescaling of the classical action is a transformation which is a universal symmetry at the classical level, but turns out to be lost when one goes to the quantum domain due to the presence of \hbar setting a scale for the action. Along this direction we have also analyzed the possibility that the \mathcal{Q}_s -symmetry can be interpreted as an extension to the phase space $\widetilde{\mathcal{M}}$ of a transformation which rescales the action already at the level of the ordinary phase space \mathcal{M} . Actually this transformation does exist (and we called it “MSA”), but it is not a “true” symmetry. The reason why we called the \mathcal{Q}_s -transformation a symmetry was that it leaves invariant the equations of motion of the system at hand. This is obviously true provided that the Lagrange equations are mapped by the transformation (\mathcal{Q}_s in this case) to new Lagrange equations. This is no longer true in the case of the MSA transformation, which is anholonomic (it transforms coordinates and differentials in an independent manner) and therefore does not preserve the Lagrange equations. However we hope to shed some further light in this direction in the future, because we

think that the rescaling of the action is something which is playing a crucial role in passing from the classical to the quantum regime of a physical system.

In the last chapter of the thesis we have studied a simpler kind of rescaling, which is known as “superconformal transformation”. In the literature this name is used to denote a composition of a supersymmetry plus a conformal transformation. In this thesis we have studied a new kind of superconformal algebra obtained by applying the CPI-formalism to a model (known in the literature as “Conformal Mechanics”) which exhibits a nonrelativistic conformal invariance. The universal susy of the CPI combines with the generators of the conformal algebra leading to a new kind of superconformal algebra. The main difference between the latter and the other models present in the literature is that in our superconformal algebra the even part is represented faithfully (and not spinorially) on the odd part. We hope that this particular model may be a useful playground to tackle the more general problem of the rescaling induced by \mathcal{Q}_S , but this still remains an open question.

Appendix A

Local Susy: Mathematical Details

A.1 Susy and time-derivative

In deriving Eqs. (2.1.4) and (2.1.5), or even in checking the global symmetry under Q_H , we had to work out things involving the variation of the kinetic piece of $\tilde{\mathcal{L}}$, i.e.:

$$[\epsilon Q_H, \lambda_a \dot{\varphi}^a + i \bar{c}_a \dot{c}^a] = (\delta \lambda_a) \dot{\varphi}^a + \lambda_a \frac{d}{dt}(\delta \varphi^a) + i(\delta \bar{c}_a) \dot{c}^a + i \bar{c}_a \frac{d}{dt}(\delta c^a) \quad (\text{A.1.1})$$

In this step we have interchanged the variation “ δ ” with the time derivative $\frac{d}{dt}$. If we actually do the time derivative of a variation (for example of φ^a), we get

$$\begin{aligned} \frac{d}{dt}(\delta \varphi^a) &= \frac{d}{dt}[\epsilon Q_H, \varphi^a] \\ &= [\frac{d}{dt}(\epsilon Q_H), \varphi^a] + [\epsilon Q_H, \frac{d\varphi^a}{dt}] \\ &= [\epsilon Q_H, \frac{d\varphi^a}{dt}] \\ &= \delta \left(\frac{d\varphi^a}{dt} \right), \end{aligned} \quad (\text{A.1.2})$$

and if ϵ is a global parameter the third equality in the equation above holds (and as a result we can interchange the variation with the time derivative) only if we use the fact that $\frac{dQ_H}{dt} = 0$. We have supposed the same thing in the case of the local variations (2.1.4)(2.1.5), and the only extra term appearing with respect to Eq.(A.1.2) is the one containing the $\dot{\epsilon}$. Using the conservation of Q_H means that we have assumed that the equations of motion hold. Actually it is better not to assume that. In fact, if we make this assumption, then the $\tilde{\mathcal{L}}$ itself, of which we are checking the invariance via the variations above, would be zero. This is due to the fact that $\tilde{\mathcal{L}}$ is proportional to the equations of motion and checking the invariance of something that is zero is a nonsense.

To avoid that problem the trick to use is to define the following integrated charge:

$$\widetilde{Q}_H = \int_0^T Q_H(t) dt, \quad (\text{A.1.3})$$

where 0 and T are the endpoints of the interval over which we consider our motion.

It is then easy to check that all the steps done in Eq.(A.1.2), once we replace Q_H with \widetilde{Q}_H , can go through without assuming the conservation of Q_H . In fact the $\frac{d\widetilde{Q}_H}{dt}$ in the third step of equation (A.1.2) is zero not because of the conservation of Q_H but because \widetilde{Q}_H is independent of t . Moreover the variation δ generated by the \widetilde{Q}_H is the same as the one generated by Q_H . This is so because in checking the variation induced by \widetilde{Q}_H we have to use the non-equal-time commutators given by the path-integral (1.1.7) which are:

$$[\varphi^a(t), \lambda_b(t')] = i\delta_b^a \delta(t - t') \quad (\text{A.1.4})$$

and similarly for the c^a and \bar{c}_a .

This charge was introduced before in the literature [9] in order to handle things in an abstract “loop space”. In our case we need that charge for the much simpler reasons explained above.

A.2 Combination of two Susy transformations

In this Appendix we will show what happens when we combine two Susy transformations.

Let us define the following two transformations $G_{\epsilon_1}, G_{\epsilon_2}$ on any of the variables $(\varphi^a, c^a, \lambda_a, \bar{c}_a)$ (which we will collectively indicate with O).

$$\begin{aligned} \delta_1 O &\equiv [G_{\epsilon_1}, O] \equiv [\bar{\epsilon}_1 \bar{Q}_H + \epsilon_1 Q_H, O] \\ \delta_2 O &\equiv [G_{\epsilon_2}, O] \equiv [\bar{\epsilon}_2 \bar{Q}_H + \epsilon_2 Q_H, O] \end{aligned} \quad (\text{A.2.1})$$

where the infinitesimal parameters ϵ_1, ϵ_2 are time dependent.

Combining two of these transformations we get

$$[\delta_1, \delta_2]O = [G_{\epsilon_1}, [G_{\epsilon_2}, O]] - [G_{\epsilon_2}, [G_{\epsilon_1}, O]]. \quad (\text{A.2.2})$$

Applying the Jacobi identity to the RHS of (A.2.2), we get

$$[\delta_1, \delta_2]O = [[G_{\epsilon_1}, G_{\epsilon_2}], O]. \quad (\text{A.2.3})$$

It is easy to find out that $[G_{\epsilon_1}, G_{\epsilon_2}]$ is given by:

$$[G_{\epsilon_1}, G_{\epsilon_2}] = 2i\beta(\bar{\epsilon}_1\epsilon_2 + \epsilon_1\bar{\epsilon}_2)\widetilde{\mathcal{H}}; \quad (\text{A.2.4})$$

inserting (A.2.4) in (A.2.3), we get

$$\begin{aligned} [\delta_1, \delta_2]O &= 2i\beta(\bar{\epsilon}_1\epsilon_2 + \epsilon_1\bar{\epsilon}_2)[\tilde{\mathcal{H}}, O] \\ &= 2\beta(\bar{\epsilon}_1\epsilon_2 + \epsilon_1\bar{\epsilon}_2)\frac{dO}{dt}. \end{aligned} \quad (\text{A.2.5})$$

So we see from here that the composition of two local Susy transformations produces a local time-translation with parameter $\bar{\epsilon}_1\epsilon_2 + \epsilon_1\bar{\epsilon}_2$.

It is also instructive to do the composition of two *finite* Susy transformations, the first (G_1) with parameter ϵ_1 and the other (G_2) with parameter ϵ_2 . The transformed variable O' has the expression:

$$O' = e^{iG_1}e^{iG_2} O e^{-iG_2}e^{-iG_1}. \quad (\text{A.2.6})$$

Using the Baker-Hausdorff identity on the RHS above, we obtain:

$$\begin{aligned} O' &= e^{[iG_1+iG_2-\frac{1}{2}[G_1,G_2]]} O e^{[-iG_1-iG_2+\frac{1}{2}[G_1,G_2]]} \\ &= e^{i[\bar{\epsilon}_1\bar{Q}_H+\epsilon_1Q_H+\bar{\epsilon}_2\bar{Q}_H+\epsilon_2Q_H-\beta(\bar{\epsilon}_1\epsilon_2+\epsilon_1\bar{\epsilon}_2)\tilde{\mathcal{H}}]} O e^{-i[\bar{\epsilon}_1\bar{Q}_H+\epsilon_1Q_H+\bar{\epsilon}_2\bar{Q}_H+\epsilon_2Q_H-\beta(\bar{\epsilon}_1\epsilon_2+\epsilon_1\bar{\epsilon}_2)\tilde{\mathcal{H}}]} \\ &= e^{i\bar{\gamma}\bar{Q}_H+i\gamma Q_H+i\Delta t\tilde{\mathcal{H}}} O e^{-i\bar{\gamma}\bar{Q}_H-i\gamma Q_H-i\Delta t\tilde{\mathcal{H}}}, \end{aligned} \quad (\text{A.2.7})$$

where $\bar{\gamma}$, γ and Δt are respectively

$$\begin{cases} \gamma = \epsilon_1 + \epsilon_2 \\ \bar{\gamma} = \bar{\epsilon}_1 + \bar{\epsilon}_2 \\ \Delta t = -\beta(\bar{\epsilon}_1\epsilon_2 + \epsilon_1\bar{\epsilon}_2). \end{cases} \quad (\text{A.2.8})$$

So we see from Eq.(A.2.7) that the composition of two finite local Susy transformations is a local Susy plus a local time-translation.

We will write down here how the variables $(\varphi^a, c^a, \lambda_a, \bar{c}_a)$ transform under a local time translation:

$$\begin{cases} \delta_{\tilde{\mathcal{H}}}^{loc}\varphi^a = [\eta(t)\tilde{\mathcal{H}}, \varphi^a] = -i\eta \omega^{an}\partial_n\tilde{\mathcal{H}} \\ \delta_{\tilde{\mathcal{H}}}^{loc}\lambda_a = [\eta(t)\tilde{\mathcal{H}}, \lambda_a] = i\eta \partial_a\tilde{\mathcal{H}} \\ \delta_{\tilde{\mathcal{H}}}^{loc}c^a = [\eta(t)\tilde{\mathcal{H}}, c^a] = -i\eta \omega^{an}\partial_n\partial_l H c^l \\ \delta_{\tilde{\mathcal{H}}}^{loc}\bar{c}_a = [\eta(t)\tilde{\mathcal{H}}, \bar{c}_a] = i\eta \bar{c}_m\omega^{mn}\partial_n\partial_a H. \end{cases} \quad (\text{A.2.9})$$

A.3 First and Second class constraints

In this Appendix we analyze the question of how the transformations (2.1.14) are generated by our first class constraints (2.1.15)(2.1.20). This is a delicate issue which is explained in detail on page 75 ff of Ref. [37]. In fact, naively the

transformation on g contained in (2.1.14) apparently cannot be obtained by doing the commutator of g with the proper gauge generators. The authors of Ref. [37] are aware of similar problems and they suggested the following approach. First let us build an extended action defined in the following way:

$$S_{ext.} = \int dt [\tilde{\mathcal{L}}_{Susy} + \Pi_\psi \dot{\psi} + \Pi_{\bar{\psi}} \dot{\bar{\psi}} + \Pi_g \dot{g} - U^{(i)} G_i] \quad (\text{A.3.1})$$

where the G_i are all the six first class constraints (2.1.24) (and not just the primary ones) and the $U^{(i)}$ the relative Lagrange multipliers. A general gauge transformation of an observable O will be:

$$\delta O = [\bar{\epsilon} \bar{Q}_H + \epsilon Q_H + \eta \tilde{\mathcal{H}} + \bar{\alpha} \Pi_\psi + \alpha \Pi_{\bar{\psi}} + \beta \Pi_g, O] \quad (\text{A.3.2})$$

where $(\bar{\epsilon}, \epsilon, \eta, \bar{\alpha}, \alpha, \beta)$ are six infinitesimal gauge parameters associated to the six generators G_i . If we consider the Lagrange multipliers U^i as functions of the basic variables, then they will also change under the gauge transformation above. As we do not know the exact expression of the U^i in terms of the basic variables, we will formally indicate their gauge variation as δU^i . Using this notation it is then a simple but long calculation to show that the gauge variation of the action $S_{ext.}$ is:

$$\begin{aligned} \delta S_{ext} = \int dt \bigg[& i\dot{\epsilon} Q_H - i\dot{\bar{\epsilon}} \bar{Q}_H - i\dot{\eta} \tilde{\mathcal{H}} - 2i\tilde{\mathcal{H}}(\epsilon\psi + \bar{\epsilon}\bar{\psi}) + \\ & + \bar{\alpha} \bar{Q}_H + \alpha Q_H - i\beta \tilde{\mathcal{H}} + \Pi_\psi \dot{\bar{\alpha}} + \Pi_{\bar{\psi}} \dot{\alpha} - i\Pi_g \dot{\beta} + \\ & - \delta U^{(2)} \Pi_\psi - \delta U^{(1)} \Pi_{\bar{\psi}} - \delta U^{(3)} \Pi_g - \delta U^{(4)} Q_H + \\ & - \delta U^{(5)} \bar{Q}_H - \delta U^{(6)} \tilde{\mathcal{H}} - U^{(4)} \bar{\epsilon} 2i\tilde{\mathcal{H}} - U^{(5)} \epsilon 2i\tilde{\mathcal{H}} \bigg], \end{aligned} \quad (\text{A.3.3})$$

where we have indicated with $U^{(1)}$, for example, the Lagrange multiplier associated to the first of the constraints in (2.1.24), with $U^{(2)}$ the one associated to the second and so on.

It is now easy to choose the variation of the Lagrange multipliers in such a way to make $\delta S_{ext} = 0$:

$$\begin{cases} \delta U^{(1)} = -\dot{\alpha} \\ \delta U^{(2)} = -\dot{\bar{\alpha}} \\ \delta U^{(3)} = -i\dot{\beta}; \end{cases} \quad \begin{cases} \delta U^{(4)} = \alpha - i\dot{\epsilon} \\ \delta U^{(5)} = \bar{\alpha} - i\dot{\bar{\epsilon}} \\ \delta U^{(6)} = -i\dot{\eta} - i\beta - 2i(\bar{\epsilon}\bar{\psi} + \epsilon\psi) + 2i(\bar{\epsilon}U^{(4)} + \epsilon U^{(5)}). \end{cases} \quad (\text{A.3.4})$$

We can now proceed as in Ref. [37] by restricting the Lagrange multipliers to be only those of the primary constraints (2.1.15)

$$U^{(4)} = U^{(5)} = U^{(6)} = 0 \quad (\text{A.3.5})$$

which imply that also the gauge variations of these variables must be zero. From these two conditions we get from Eq.(A.3.4) the following relations among the six gauge parameters

$$\begin{cases} \alpha = i\dot{\epsilon} \\ \bar{\alpha} = i\dot{\bar{\epsilon}} \\ \beta = -\dot{\eta} - 2(\bar{\epsilon}\bar{\psi} + \epsilon\psi). \end{cases} \quad (\text{A.3.6})$$

As a consequence the general gauge variation of an observable O given in Eq.(A.3.2) becomes

$$\delta O = [\bar{\epsilon}\bar{Q}_H + \epsilon Q_H + \eta\tilde{\mathcal{H}} + i\dot{\bar{\epsilon}}\Pi_\psi + i\dot{\epsilon}\Pi_{\bar{\psi}} + (-\dot{\eta} - 2(\bar{\epsilon}\bar{\psi} + \epsilon\psi))\Pi_g, O] \quad (\text{A.3.7})$$

and applying it on the three variables $(\psi, \bar{\psi}, g)$ we get:

$$\begin{cases} \delta\psi = [i\dot{\bar{\epsilon}}\Pi_\psi, \psi] = i\dot{\bar{\epsilon}}\psi \\ \delta\bar{\psi} = [i\dot{\epsilon}\Pi_{\bar{\psi}}, \bar{\psi}] = i\dot{\epsilon}\bar{\psi} \\ \delta g = [-(\dot{\eta} + 2(\bar{\epsilon}\bar{\psi} + \epsilon\psi))\Pi_g, g] = i\dot{\eta}g + 2i(\bar{\epsilon}\bar{\psi} + \epsilon\psi)g. \end{cases} \quad (\text{A.3.8})$$

This is exactly the transformation (2.1.14) obtained here from the generators G_i of Eq.(2.1.24). This concludes the explanation of how the variations (2.1.14), derived from a pure Lagrangian variation, could be obtained via the *canonical* gauge generators G_i .

A.4 How to gauge away $(\psi, \bar{\psi}, g)$

In this Appendix, for purely pedagogical reasons, we will show how to gauge away the $(\psi, \bar{\psi}, g)$.

The infinitesimal transformations are given in Eq.(2.1.14) and the first thing to do is to build finite transformations out of the infinitesimal ones. If we start from a configuration $(\psi_0(t), \bar{\psi}_0(t), g_0(t))$, after one step we arrive at $(\psi_1(t), \bar{\psi}_1(t), g_1(t))$ which are given by:

$$\begin{cases} \psi_1(t) = \psi_0(t) + i\dot{\bar{\epsilon}}(t)\psi_0(t) \\ \bar{\psi}_1(t) = \bar{\psi}_0(t) + i\dot{\epsilon}(t)\bar{\psi}_0(t) \\ g_1(t) = g_0(t) + i\dot{\eta}(t)g_0(t) + 2i(\epsilon(t)\psi_0(t) + \bar{\epsilon}(t)\bar{\psi}_0(t))g_0(t). \end{cases} \quad (\text{A.4.1})$$

It is not difficult to work out what we get after N steps:

$$\begin{cases} \psi_N(t) = \psi_0(t) + iN\dot{\bar{\epsilon}}(t)\psi_0(t) \\ \bar{\psi}_N(t) = \bar{\psi}_0(t) + iN\dot{\epsilon}(t)\bar{\psi}_0(t) \\ g_N(t) = g_0(t) + iN\dot{\eta}(t)g_0(t) + 2i(N\epsilon(t)\psi_0(t) + N\bar{\epsilon}(t)\bar{\psi}_0(t))g_0(t) - N(N+1)(\epsilon\dot{\bar{\epsilon}} + \bar{\epsilon}\dot{\epsilon}). \end{cases} \quad (\text{A.4.2})$$

Taking now the limit $N \rightarrow \infty$, but with the conditions:

$$\begin{cases} N\epsilon(t) \longrightarrow \Delta(t) \\ N\bar{\epsilon}(t) \longrightarrow \bar{\Delta}(t) \\ N\eta(t) \longrightarrow \Delta_g(t), \end{cases} \quad (\text{A.4.3})$$

where the various $\Delta(t)$ are finite quantities, we get that a finite transformation has the form:

$$\begin{cases} \psi(t) = \psi_0(t) + i\dot{\bar{\Delta}}(t) \\ \bar{\psi}(t) = \bar{\psi}_0(t) + i\dot{\Delta}(t) \\ g(t) = g_0(t) + i\dot{\Delta}_g(t) + 2i(\Delta(t)\psi_0(t) + \bar{\Delta}(t)\bar{\psi}_0(t)) - (\Delta\dot{\bar{\Delta}} + \bar{\Delta}\dot{\Delta}). \end{cases} \quad (\text{A.4.4})$$

From the equation above it is easy to see that with the following choice of Δ 's

$$\begin{cases} \bar{\Delta}(t) = i \int_0^t \psi_0(\tau) d\tau \\ \Delta(t) = i \int_0^t \bar{\psi}_0(\tau) d\tau \\ \Delta_g(t) = \int_0^t d\tau [ig_0(\tau) - 2(\Delta\psi_0 + \bar{\Delta}\bar{\psi}_0) - i(\Delta\dot{\bar{\Delta}} + \bar{\Delta}\dot{\Delta})] \end{cases} \quad (\text{A.4.5})$$

we can bring the $(\psi, \bar{\psi}, g)$ to zero. We should anyhow be careful and see whether there are no obstructions to this construction. Actually, after Eq.(2.1.5) we said that, in order that the surface terms disappear, we needed to require that $\epsilon(t)$ and $\bar{\epsilon}(t)$ be zero at the end-points $(0, T)$ of integration. From Eq.(A.4.3) one sees that this implies:

$$\begin{aligned} \Delta(0) &= \bar{\Delta}(0) = 0 \\ \Delta(T) &= \bar{\Delta}(T) = 0. \end{aligned} \quad (\text{A.4.6})$$

While the first condition is easily satisfied, as can be seen from Eq.(A.4.5), the second one would imply:

$$\begin{aligned} \Delta(T) &= i \int_0^T \bar{\psi}_0(\tau) d\tau = 0 \\ \bar{\Delta}(T) &= i \int_0^T \psi_0(\tau) d\tau = 0. \end{aligned} \quad (\text{A.4.7})$$

This is a condition which is not satisfied by any initial configuration $\psi_0, \bar{\psi}_0$ but only by special ones. So we can say that, if we want transformations which do not leave surface terms, then it may be impossible to gauge away $(\psi, \bar{\psi})$. Not to have surface terms may turn out to be an important issue in some contexts. Anyhow this problem does not arise in the time-reparametrization transformation because, as we see from Eq.(2.1.11), that transformation does not generate surface terms.

A.5 Q_{BRS} , \overline{Q}_{BRS} and Physical States

In this Appendix we will show how the constraints (2.2.13) affect the Hilbert space of the system. We know that the *physical* states should be annihilated by the constraints:

$$\begin{cases} [\Pi_\alpha + Q_{BRS}] | \text{phys} \rangle = 0 \\ [\Pi_{\overline{\alpha}} + \overline{Q}_{BRS}] | \text{phys} \rangle = 0 \end{cases} \quad (\text{A.5.1})$$

and so this seems to restrict the original Hilbert space of the system. On the other hand we have proved that the system obeying these constraints and with Lagrangian (2.2.11) has the same number of degrees of freedom as the original system with Lagrangian (1.1.6) and moreover they seem equivalent. If that is so then the Hilbert space of the *physical* states should be equivalent or isomorphic to the original Hilbert space. This is what we are going to prove in what follows.

Let us first solve the constraint (A.5.1). The wave-functions $\Psi(\dots)$ of the system will depend not only on the (φ^a, c^a) but also on the gauge-parameters $\alpha(t)$ and $\overline{\alpha}(t)$. So equation (A.5.1) takes the form:

$$\begin{cases} \frac{\partial \Psi(\varphi, c; \alpha, \overline{\alpha})}{\partial \alpha} = -Q_{BRS} \Psi(\varphi, c; \alpha, \overline{\alpha}) \\ \frac{\partial \Psi(\varphi, c; \alpha, \overline{\alpha})}{\partial \overline{\alpha}} = -\overline{Q}_{BRS} \Psi(\varphi, c; \alpha, \overline{\alpha}) \end{cases} \quad (\text{A.5.2})$$

whose solution is

$$\Psi(\varphi, c; \alpha, \overline{\alpha}) = \exp[-\alpha Q_{BRS} - \overline{\alpha} \overline{Q}_{BRS}] \psi(\varphi, c) \quad (\text{A.5.3})$$

where $\psi(\varphi, c)$ is a state of the Hilbert space of the old system with Lagrangian (1.1.6) and the Q_{BRS} and \overline{Q}_{BRS} should be interpreted as the differential operator associated to the relative charges via the substitutions (1.3.20) and (1.3.25); the same for all the Hamiltonians which we will use from now on. To prove that the two systems are equivalent we should prove that there is an isomorphism in Hilbert space between the solutions of the two Koopman-von Neumann¹ equations, the first one associated to the old Hamiltonian (1.1.8) and the second to the Hamiltonian of the Lagrangian (2.2.11). This last one is the *primary* [22] Hamiltonian:

$$\tilde{\mathcal{H}}_P \equiv \tilde{\mathcal{H}}_{can} + \mu(\Pi_\alpha + Q_{BRS}) + \overline{\mu}(\Pi_{\overline{\alpha}} + \overline{Q}_{BRS}) \quad (\text{A.5.4})$$

where μ and $\overline{\mu}$ are Lagrange multipliers and $\tilde{\mathcal{H}}_{can}$ is the *canonical* [22] Hamiltonian associated to the Lagrangian (2.2.11).

The Koopman-von Neumann equation for this system is:

¹By Koopman von Neumann equation we mean the analog of the Liouville equations built via the full \mathcal{H} and not via just its bosonic part.

$$\tilde{\mathcal{H}}_P \Psi(\varphi, c, t; \alpha, \bar{\alpha}) = i \frac{\partial \Psi(\varphi, c, t; \alpha, \bar{\alpha})}{\partial t} \quad (\text{A.5.5})$$

which can be rewritten as:

$$[\tilde{\mathcal{H}} + \mu(\Pi_\alpha + Q_{BRS}) + \bar{\mu}(\Pi_{\bar{\alpha}} + \bar{Q}_{BRS})] \Psi(\varphi, c, t; \alpha, \bar{\alpha}) = i \frac{\partial \Psi(\varphi, c, t; \alpha, \bar{\alpha})}{\partial t}. \quad (\text{A.5.6})$$

Since $\Psi(\varphi, c, t; \alpha, \bar{\alpha})$ is annihilated by the constraints (A.5.1) we get:

$$\tilde{\mathcal{H}} \Psi(\varphi, c, t; \alpha, \bar{\alpha}) = i \frac{\partial \Psi(\varphi, c, t; \alpha, \bar{\alpha})}{\partial t}; \quad (\text{A.5.7})$$

now we use (A.5.3) in (A.5.7) and this yields:

$$\tilde{\mathcal{H}} \exp[-\alpha Q_{BRS} - \bar{\alpha} \bar{Q}_{BRS}] \psi(\varphi, c, t) = i \frac{\partial}{\partial t} [\exp(-\alpha Q_{BRS} - \bar{\alpha} \bar{Q}_{BRS}) \psi(\varphi, c, t)]. \quad (\text{A.5.8})$$

The last step is to work out the derivatives in Eq.(A.5.8); we obtain:

$$\exp[-\alpha Q_{BRS} - \bar{\alpha} \bar{Q}_{BRS}] \tilde{\mathcal{H}} \psi(\varphi, c, t) = \exp[-\alpha Q_{BRS} - \bar{\alpha} \bar{Q}_{BRS}] i \frac{\partial \psi(\varphi, c, t)}{\partial t} \quad (\text{A.5.9})$$

which holds if and only if:

$$\tilde{\mathcal{H}} \psi = i \frac{\partial \psi}{\partial t} \quad (\text{A.5.10})$$

and this concludes the proof that the two systems have not only the same number of degrees of freedom but also the same Hilbert space.

A.6 Details about Eq.(2.2.34)

In this Appendix we provide details regarding the derivation of Eq.(2.2.34). Consider first the dependence on g . From

$$|\text{phys}\rangle = Q_{(1)}|\chi\rangle \quad \text{and} \quad \Pi_g |\text{phys}\rangle = 0 \quad (\text{A.6.1})$$

we infer that

$$\Pi_g Q_{(1)}|\chi\rangle = Q_{(1)}\Pi_g|\chi\rangle = 0, \quad (\text{A.6.2})$$

which means that

$$\Pi_g|\chi\rangle \in \ker Q_{(1)}. \quad (\text{A.6.3})$$

If we represent Π_g as $\Pi_g = -i\frac{\partial}{\partial g}$, Eq.(A.6.3) implies:

$$\frac{\partial}{\partial g}|\chi\rangle = \sum_m f_m(g, \alpha)|\zeta_m\rangle \quad (\text{A.6.4})$$

where $|\zeta_m\rangle$ form a basis of $\ker Q_{(1)}$. Solving this last differential equation we get

$$|\chi\rangle = |\chi_0; \alpha\rangle + \sum_m \left[\int dg f_m(g, \alpha) \right] |\zeta_m\rangle \quad (\text{A.6.5})$$

where $|\chi_0; \alpha\rangle$ does not depend on g anymore. By the same line of reasoning we can prove that $\Pi_\alpha|\chi\rangle \in \ker Q_{(1)}$, which in turn implies that $\Pi_\alpha|\chi_0; \alpha\rangle \in \ker Q_{(1)}$. We can repeat the previous steps and we arrive at the relation:

$$|\chi_0; \alpha\rangle = |\chi_0\rangle + \sum_m \left[\int d\alpha l_m(g, \alpha) \right] |\zeta_m\rangle \quad (\text{A.6.6})$$

which, substituted in Eq. (A.6.5), yields:

$$|\chi\rangle = |\chi_0\rangle + \sum_m \left[\int d\alpha l_m(g, \alpha) \right] |\zeta_m\rangle + \sum_m \left[\int dg f_m(g, \alpha) \right] |\zeta_m\rangle \equiv |\chi_0\rangle + |\zeta; \alpha, g\rangle, \quad (\text{A.6.7})$$

as we claimed in Eq. (2.2.34).

Appendix B

CPI and κ -symmetry: Mathematical Details

In this Appendix we show the standard procedure to build the Dirac Brackets for the 2nd-class constraints listed in Eq.(3.1.7). They are explicitly:

$$\left\{ \begin{array}{ll} \Pi_p^\mu = 0 & (a) \\ (\Pi_x)_\mu - p_\mu = 0 & (b) \\ D^a \equiv (\Pi_{\bar{\zeta}})^a + \frac{i}{2}(\not{x}\zeta)^a = 0 & (c) \\ \overline{D}_a \equiv (\Pi_\zeta)_a + \frac{i}{2}(\bar{\zeta}\not{x})_a = 0 & (d), \end{array} \right. \quad (\text{B.0.1})$$

and we can arrange them in the following row:

$$\phi_i = (\Pi_p^\mu, (\Pi_x)_\mu - p_\mu, D^a, \overline{D}_a). \quad (\text{B.0.2})$$

Then, in order to construct the Dirac Brackets associated to these constraints, we need the matrix Δ_{ij} defined as follows:

$$\Delta_{ij} = [\phi_i, \phi_j]_{PB}. \quad (\text{B.0.3})$$

The Poisson Brackets among all the constraints are found after a long but easy calculation:

$$[(\Pi_p)^\mu, (\Pi_x)_\nu - p_\nu]_{PB} = \delta_\nu^\mu \quad (\text{B.0.4})$$

$$[(\Pi_p)^\mu, D^a]_{PB} = -\frac{i}{2}(\gamma^\mu)_b^a \zeta^b \quad (\text{B.0.5})$$

$$[(\Pi_p)^\mu, \bar{D}_b]_{PB} = -\frac{i}{2}\bar{\zeta}_d(\gamma^\mu)_b^d \quad (\text{B.0.6})$$

$$[(\Pi_x)_\nu - p_\nu, (\Pi_p)^\mu]_{PB} = -\delta_\nu^\mu \quad (\text{B.0.7})$$

$$[(\Pi_x)_\nu - p_\nu, D^a]_{PB} = 0 \quad (\text{B.0.8})$$

$$[(\Pi_x)_\nu - p_\nu, \bar{D}_b]_{PB} = 0 \quad (\text{B.0.9})$$

$$[D^a, (\Pi_p)^\mu]_{PB} = \frac{i}{2}(\gamma^\mu)_b^a \zeta^b \quad (\text{B.0.10})$$

$$[D^a, (\Pi_x)_\nu - p_\nu]_{PB} = 0 \quad (\text{B.0.11})$$

$$[D^a, \bar{D}_b]_{PB} = i(\not{x})_b^a \quad (\text{B.0.12})$$

$$[\bar{D}_b, (\Pi_p)^\mu]_{PB} = \frac{i}{2}\bar{\zeta}_d(\gamma^\mu)_b^d \quad (\text{B.0.13})$$

$$[\bar{D}_b, (\Pi_x)_\nu - p_\nu]_{PB} = 0 \quad (\text{B.0.14})$$

$$[\bar{D}_b, D^a]_{PB} = i(\not{x})_b^a = i(\not{x}^T)_b^a \quad (\text{B.0.15})$$

and therefore the matrix Δ_{ij} takes the form:

$$\Delta_{ij} = [\phi_i, \phi_j]_{PB} = \begin{pmatrix} 0 & 1 & -\frac{i}{2}(\gamma^\mu \zeta)^T & -\frac{i}{2}\bar{\zeta} \gamma^\mu \\ -1 & 0 & 0 & 0 \\ \frac{i}{2}\gamma^\mu \zeta & 0 & 0 & i\not{x} \\ \frac{i}{2}(\bar{\zeta} \gamma^\mu)^T & 0 & i\not{x}^T & 0 \end{pmatrix}. \quad (\text{B.0.16})$$

This is an even supermatrix and there are many ways to find the inverse. After a little algebra one finds that the inverse has the following form:

$$(\Delta^{-1})^{ij} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & \frac{1}{2}\bar{\zeta} \gamma^\mu \not{x}^{-1} & \frac{1}{2}(\not{x}^{-1} \gamma^\mu \zeta)^T \\ 0 & \frac{1}{2}(\bar{\zeta} \gamma^\mu \not{x}^{-1})^T & 0 & -i(\not{x}^T)^{-1} \\ 0 & \frac{1}{2}\not{x}^{-1} \gamma^\mu \zeta & -i\not{x}^{-1} & 0 \end{pmatrix}. \quad (\text{B.0.17})$$

As a consequence, the Dirac Brackets derived from the previous constraints are:

$$\begin{aligned}
[A, B]_{DB} = & [A, B]_{PB} + \\
& - [A, (\Pi_p)^\mu]_{PB} (-\delta_\nu^\mu) [(\Pi_x)_\nu - p_\nu, B]_{PB} + \\
& - [A, (\Pi_x)_\nu - p_\nu]_{PB} \delta_\mu^\nu [(\Pi_p)^\mu, B]_{PB} + \\
& - [A, (\Pi_x)_\nu - p_\nu]_{PB} \frac{1}{2} (\bar{\zeta} \gamma^\nu \not{x}^{-1})_a [D^a, B]_{PB} + \\
& - [A, (\Pi_x)_\nu - p_\nu]_{PB} \frac{1}{2} (\not{x}^{-1} \gamma^\mu \zeta)_b^T [\bar{D}_b, B]_{PB} + \\
& - [A, D^a]_{PB} \frac{1}{2} [(\bar{\zeta} \gamma^\mu \not{x}^{-1})^T]^a [(\Pi_x)_\nu - p_\nu, B]_{PB} + \\
& - [A, D^a]_{PB} (-i) [(\not{x}^T)^{-1}]_b^a [\bar{D}_b, B] + \\
& - [A, \bar{D}_b]_{PB} \frac{1}{2} (\not{x}^{-1} \gamma^\mu \zeta)^b [(\Pi_x)_\nu - p_\nu, B]_{PB} + \\
& - [A, \bar{D}_b]_{PB} (-i) (\not{x}^{-1})_a^b [D^a, B];
\end{aligned} \tag{B.0.18}$$

and it is not a difficult exercise (even if rather long) to check that the previous expression leads to the same brackets as those in Eqs.(3.1.11)-(3.1.17).

Appendix C

The Conformal Group

In this Appendix we give a brief review of the conformal group and its representations in various dimensions. We think this can be useful because of the big interest that this group has recently gained in many fields of theoretical physics, from astrophysics to statistical physics.

C.1 Conformal Transformations

Consider a Riemannian (or Lorentzian¹) manifold \mathcal{M} with metric g ; it is well known that the change of the metric under a transformation of coordinates is given by:

$$\begin{cases} x \longrightarrow y(x) \\ g_{\mu\nu}(x) \longrightarrow g'_{\mu\nu}(y) = \frac{\partial x^\alpha}{\partial y^\mu} \frac{\partial x^\beta}{\partial y^\nu} g_{\alpha\beta}(x). \end{cases} \quad (\text{C.1.1})$$

A transformation is called *conformal* if Eq.(C.1.1) takes the form:

$$g'_{\mu\nu}(y) = \Omega(x) g_{\mu\nu}(x). \quad (\text{C.1.2})$$

Now we want to analyze the form which the factor $\Omega(x)$ can have in order to find out the generators of the group. First of all we consider the infinitesimal version of (C.1.1):

$$\begin{cases} y^\mu = x^\mu + \epsilon^\mu(x) \\ g'_{\mu\nu}(y) = g_{\mu\nu}(x) - g_{\mu\beta}(x) \partial_\nu \epsilon^\beta - g_{\alpha\nu}(x) \partial_\mu \epsilon^\alpha; \end{cases} \quad (\text{C.1.3})$$

and if we write $\Omega(x) \equiv 1 - \omega(x)$ (according to the infinitesimal character of the

¹A metric g is called “Riemannian” if its signature is n (n being the dimension of the manifold, while it is called “Lorentzian” if the signature is $n - 2$).

transformation), from (C.1.2) and (C.1.3) we obtain:

$$\omega(x)g_{\mu\nu}(x) = g_{\mu\beta}(x)\partial_\nu\epsilon^\beta + g_{\alpha\nu}(x)\partial_\mu\epsilon^\alpha. \quad (\text{C.1.4})$$

Consider for simplicity the case $g_{\mu\nu} = \eta_{\mu\nu}$: the previous equation becomes:

$$\omega(x)\eta_{\mu\nu} = \partial_\nu\epsilon_\mu(x) + \partial_\mu\epsilon_\nu(x), \quad (\text{C.1.5})$$

from which it is immediately found that:

$$\omega(x) = \frac{2}{d}\partial_\mu\epsilon^\mu \quad (\text{C.1.6})$$

where d is the dimension of the space-time. If we apply ∂_ρ to Eq.(C.1.5) we get:

$$\partial_\rho\omega(x)\eta_{\mu\nu} = \partial_{\rho\nu}^2\epsilon_\mu(x) + \partial_{\rho\mu}^2\epsilon_\nu(x), \quad (\text{C.1.7})$$

from which, after some rearrangements and permutations of the indices, we arrive at:

$$2\partial_\mu\partial_\nu\epsilon_\rho(x) = \eta_{\mu\rho}\partial_\nu\omega(x) + \eta_{\nu\rho}\partial_\mu\omega(x) - \eta_{\mu\nu}\partial_\rho\omega(x). \quad (\text{C.1.8})$$

Now we can multiply the previous equation by $\eta^{\mu\nu}$ and then take a differentiation ∂_σ ; what we get is:

$$2\partial^2\partial_\sigma\epsilon_\rho(x) = (2-d)\partial_\rho\partial_\sigma\omega(x), \quad (\text{C.1.9})$$

which, since the RHS is symmetric under exchange $\rho \longleftrightarrow \sigma$, can be rewritten as:

$$\partial^2(\partial_\sigma\epsilon_\rho + \partial_\rho\epsilon_\sigma)(x) = (2-d)\partial_\rho\partial_\sigma\omega(x). \quad (\text{C.1.10})$$

Now we want to compare Eq.(C.1.9) with the Laplacian ∂^2 of Eq.(C.1.5) which is:

$$\partial^2(\partial_\sigma\epsilon_\rho + \partial_\rho\epsilon_\sigma)(x) = \eta_{\sigma\rho}\partial^2\omega(x); \quad (\text{C.1.11})$$

from the comparison between (C.1.11) with (C.1.10) we finally get the equations which characterize the parameter ω of the an infinitesimal conformal transformation:

$$\begin{aligned} (2-d)\partial_\rho\partial_\sigma\omega(x) &= \eta_{\sigma\rho}\partial^2\omega(x) & (a) \\ (1-d)\partial^2\omega(x) &= 0 & (b) \\ \text{where } \omega(x) &= \frac{2}{d}\partial_\mu\epsilon^\mu(x). \end{aligned}$$

(C.1.12)

We can now distinguish three cases: $d = 1$, $d > 2$ and $d = 2$.

C.1.1 Case $d = 1$

This is obviously the simplest case. In fact from Eq.(C.1.12) it is clear that no constraint is imposed on $\omega(x)$ and therefore on $\epsilon(x)$. As we could guess, we deduce that in $d = 1$ every coordinate transformation is conformal.

C.1.2 Case $d > 2$

In this case we have that from (C.1.12-a) and (C.1.12-b) we obtain:

$$\partial_\mu \partial_\nu \omega(x) = 0, \quad (\text{C.1.13})$$

which implies that $\omega(x)$ can be at most a linear function of x :

$$\omega(x) = A + B_\mu x^\mu, \quad (\text{C.1.14})$$

with A and B_μ are independent of x . Next, as a consequence of Eq.(C.1.6) we have that $\epsilon_\mu(x)$ can be at most a quadratic function of x :

$$\epsilon_\mu(x) = a_\mu + b_{\mu\nu} x^\nu + c_{\mu\nu\rho} x^\nu x^\rho. \quad (\text{C.1.15})$$

After some long and boring steps [21] making use of some of the previous equations, we can arrive at the following parametrization of an infinitesimal conformal transformation in $d > 2$ dimensions:

$\begin{aligned} \epsilon_\mu(x) &= a_\mu + (m_{\mu\nu} + f\eta_{\mu\nu})x^\nu + 2x_\mu(h \cdot x) - h_\mu x^2 & (a) \\ \omega(x) &= f + 4h \cdot x. & (b) \\ \text{remember that } \Omega(x) &= 1 - \omega(x) \end{aligned}$	(C.1.16)
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where all the parameters a_μ , $m_{\mu\nu} = m_{\nu\mu}$, f and h_μ are independent of x and are functions of the parameters entering Eqs.(C.1.14)(C.1.15). If we exponentiate the infinitesimal transformation to get the finite one we obtain that a generic *finite* conformal transformation is a composition of the following transformations:

Conformal Group (d>2)

Translations	$(x')^\mu = x^\mu + a^\mu$	$(\Omega = 1)$	(C.1.17)
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Dilations	$(x')^\mu = \lambda x^\mu$	$(\Omega = \lambda^{-2})$	(C.1.18)
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Lorentz	$(x')^\mu = M^\mu_\nu x^\nu$	$(\Omega = 1)$	(C.1.19)
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Special Conf.	$(x')^\mu = \frac{x^\mu - h^\mu x^2}{1 - 2k \cdot x + k^2 x^2}$	$(\Omega = 1 - 2k \cdot x + k^2 x^2)$	(C.1.20)
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We can proceed and find out the generators of the previous transformations:

$$\text{Translations} \quad P_\mu = -i\partial_\mu \quad (\text{C.1.21})$$

$$\text{Dilations} \quad D = -ix^\mu\partial_\mu \quad (\text{C.1.22})$$

$$\text{Lorentz} \quad L_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu) \quad (\text{C.1.23})$$

$$\text{Special Conf.} \quad K_\mu = -i(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu) \quad (\text{C.1.24})$$

which satisfy the following algebra:

Conformal Algebra

$[D, P_\mu] = iP_\mu$	(a)	(C.1.25)
$[D, K_\mu] = -iK_\mu$	(b)	
$[K_\mu, P_\mu] = 2i(\eta_{\mu\nu}D - L_{\mu\nu})$	(c)	
$[K_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu)$	(d)	
$[P_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu)$	(e)	
$[L_{\mu\nu}, L_{\rho\sigma}] = i(\eta_{\nu\rho}L_{\mu\sigma} + \eta_{\mu\sigma}L_{\nu\rho} - \eta_{\mu\rho}L_{\nu\sigma} - \eta_{\nu\sigma}L_{\mu\rho})$	(f)	

It is easy to construct an isomorphism between the Conformal Group in d dimensions and the groups $SO(d+1, 1)$ or $SO(d, 2)$, depending on the signature of the metric $\eta_{\mu\nu}$: if $\eta_{\mu\nu}$ is Euclidean we have $Conf(d) \cong SO(d+1, 1)$, while $Conf(d) \cong SO(d, 2)$ if $\eta_{\mu\nu}$ is Lorentzian. In fact, if we define the following correspondences:

$$J_{\mu\nu} = L_{\mu\nu} \quad (\text{C.1.26})$$

$$J_{-10} = D \quad (\text{C.1.27})$$

$$J_{-1\nu} = \frac{1}{2}(P_\mu - K_\mu) \quad (\text{C.1.28})$$

$$J_{0\nu} = \frac{1}{2}(P_\mu + K_\mu) \quad (\text{C.1.29})$$

where $J_{ab} = -J_{ba}$ and $a, b \in (-1, 0, 1, 2, \dots, d)$, we easily see that the J_{ab} operators satisfy the $SO(d, 2)$ algebra:

$$[J_{ab}, J_{cd}] = i(\eta_{ad}J_{bc} + \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac}). \quad (\text{C.1.30})$$

C.1.3 Case $d = 2$

We still have to discuss the conformal transformations in two dimensions. First of all, if we consider again Eqs.(C.1.5)(C.1.6) and we set $d = 2$ we obtain the following equations:

$$\partial_\alpha \epsilon^\alpha \eta_{\mu\nu} = \partial_\nu \epsilon_\mu(x) + \partial_\mu \epsilon_\nu(x), \quad (\text{C.1.31})$$

from which:

$$\boxed{\begin{array}{ll} \mu = \nu & \longrightarrow \quad \partial_1 \epsilon_1 = \partial_2 \epsilon_2 \\ \mu \neq \nu & \longrightarrow \quad \partial_1 \epsilon_2 = -\partial_2 \epsilon_1. \end{array}} \quad (\text{C.1.32})$$

Eqs.(C.1.32) are nothing but the Cauchy-Riemann equations for ϵ_1 and ϵ_2 . This suggests to use a complex formalism for describing the transformations at hand. Thus we define:

$$\begin{aligned} z &\equiv x_1 + ix_2; \\ \bar{z} &\equiv x_1 - ix_2; \\ \epsilon &\equiv \epsilon_1 + i\epsilon_2; \end{aligned} \quad (\text{C.1.33})$$

consequently a generic conformal transformation takes the form:

$$z' = z + \epsilon(z, \bar{z}). \quad (\text{C.1.34})$$

But the Cauchy-Riemann conditions (C.1.32) impose the constraint²

$$\partial_{\bar{z}} \epsilon(z, \bar{z}) = 0, \quad (\text{C.1.35})$$

and then Eq.(C.1.34) becomes:

$$z' = z + \epsilon(z). \quad (\text{C.1.36})$$

Now suppose that $\epsilon(z)$ is smooth enough to be expanded in Laurent series around $z = 0$; in this case we can write:

$$\epsilon(z) = \sum_{n=-\infty}^{+\infty} c_n z^{n+1}, \quad (\text{C.1.37})$$

and the effect of a conformal transformation on a (for example) scalar field is given by:

$$\begin{cases} \Phi(z', \bar{z}') = \Phi(z, \bar{z}) \\ \delta \Phi(z, \bar{z}) = \sum_{n=-\infty}^{+\infty} (c_n l_n + \bar{c}_n \bar{l}_n) \Phi(z, \bar{z}), \end{cases} \quad (\text{C.1.38})$$

where we have introduced the following generators:

$$\begin{cases} l_n \equiv -z^{n+1} \partial_z \\ \bar{l}_n \equiv -\bar{z}^{n+1} \partial_{\bar{z}}. \end{cases} \quad (\text{C.1.39})$$

The algebra realized by the previous operators is easily found:

²Note that Eq.(C.1.35) is not equivalent to the analyticity of $\epsilon(z)$, because also the continuity of $\epsilon(z)$ is required.

$$\boxed{
\begin{aligned}
[l_n, l_m] &= (n - m)l_{n+m} \\
[\bar{l}_n, \bar{l}_m] &= (n - m)\bar{l}_{n+m} \\
[l_n, \bar{l}_m] &= 0.
\end{aligned}
} \tag{C.1.40}$$

This algebra is nothing but a Virasoro algebra without central charge. Therefore the important message is that while in $d = 1$ and $d > 2$ dimensions the Conformal Group has a finite number of generators, in $d = 2$ there is an infinite number of generators. Furthermore there is another point to be discussed: so far we have focused on *local* conformal transformations – *local* in the sense that we have never worried about the possibility of inverting the conformal map $z \longrightarrow z'(z)$ (and therefore we have never cared about its domain of definition). If we restrict to the conformal transformations which map the whole Riemann sphere (i.e. the complex plane plus ∞) onto itself, we obtain the *Global* Conformal Group, which is easy to show to be formed by the following transformations:

$$z' = f(z) = \frac{az + b}{cz + d} \quad \text{with: } ad - bc = 1. \tag{C.1.41}$$

It is easy to check that these transformations form a group since every $f(z)$ can be associated to a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in such a way that the composition of two conformal transformations becomes the ordinary product between matrices: $f \circ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix}$. This implies that the Global Conformal Group is isomorphic to $SL(2, C)$ which in turn is isomorphic to $SO(3, 1)$. In fact the Virasoro subalgebra which generates the Global Conformal Group is given by:

$$G : \begin{cases} l_{-1}; & \bar{l}_{-1}; \\ l_0; & \bar{l}_0; \\ l_1; & \bar{l}_1; \end{cases} \cong SO(3, 1). \tag{C.1.42}$$

From this subalgebra we can extract another subalgebra taking — for example — the “real” part of G :

$$G_{\text{real}} : \begin{cases} L_{-1} \equiv l_{-1} + \bar{l}_{-1}; \\ L_0 \equiv l_0 + \bar{l}_0; \\ L_1 \equiv l_1 + \bar{l}_1, \end{cases} \tag{C.1.43}$$

and these operators generate the so-called Conformal Group in 0+1 dimensions, which we used in Section 5.1. We can see that by using the following correspondence:

$$\begin{aligned}
L_{-1} &\longleftrightarrow iH \\
L_0 &\longleftrightarrow iD \\
L_1 &\longleftrightarrow iK;
\end{aligned} \tag{C.1.44}$$

if we compare the two algebras (that of (L_{-1}, L_0, L_1) (inherited by Eq.(C.1.40)) and that of iH, iD, iK (derived from Eqs.(5.1.10)-(5.1.12)), we see that it is precisely the same.

Appendix D

General Relativity and CME

D.1 The Reissner-Nordström metric

For our purposes we shall consider charged and static (i.e. non-rotating) black holes and, as a starting point, we write down the metric of a spherically symmetric space:

$$ds^2 = -e^{2\alpha(r,t)} dt^2 + e^{2\beta(r,t)} dr^2 + r^2 d\Omega^2, \quad (\text{D.1.1})$$

where:

$$d\Omega^2 = \sin^2 \theta d\varphi^2 + d\theta^2. \quad (\text{D.1.2})$$

As is well known, $\alpha(r, t)$ and $\beta(r, t)$ have to be determined by solving Einstein's equations. To do that, we impose that the energy-momentum tensor $T_{\mu\nu}$ be determined only by the Maxwell tensor $F_{\mu\nu}$, because we are supposing to have a point particle with charge Q and mass M at the origin of the frame. The Einstein equations then read:

$$R_{\mu\nu} - g_{\mu\nu} R = 8\pi G T_{\mu\nu} = 8\pi G \left[\frac{1}{4\pi} \left(F_{\mu\rho} F_{\nu}^{\rho} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right) \right], \quad (\text{D.1.3})$$

where $F_{\mu\nu}$ must obey the following Maxwell equations:

$$g^{\mu\nu} \nabla_{\mu} F_{\nu\rho} = 0 \quad (\text{D.1.4})$$

$$\nabla_{\mu} F_{\nu\rho} + \nabla_{\nu} F_{\rho\mu} + \nabla_{\rho} F_{\mu\nu} = 0 \quad (\text{D.1.5})$$

$$(\text{obviously: } \nabla_{\mu} V^{\alpha} \equiv \partial_{\mu} V^{\alpha} + \Gamma_{\mu\sigma}^{\alpha} V^{\sigma}). \quad (\text{D.1.6})$$

The spherical symmetry of the problem implies that the only non vanishing components are:

$$F_{tr} = -F_{rt} = -\frac{Q}{r^2}. \quad (\text{D.1.7})$$

Then, the solution of the Einstein equations is:

$$ds^2 = -\Delta dt^2 + \Delta^{-1} dr^2 + r^2 d\Omega^2 \quad (\text{D.1.8})$$

$$\text{with } \Delta = 1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}, \quad (\text{D.1.9})$$

which is just the Reissner-Nordström metric. The singularities of this metric are immediately found:

$$r = 0 \quad \Rightarrow \quad \text{essential singularity} \quad (\text{D.1.10})$$

$$\Delta = 0 \quad \Rightarrow \quad r_{\pm} = GM \pm \sqrt{G^2 M^2 - GQ^2} \quad (\text{event horizons}). \quad (\text{D.1.11})$$

The presence of one, two or no event-horizon depends on the sign of $(G^2 M^2 - GQ^2)$ (from now on we put $G = 1$ and use the so-called *geometric units*). Then we can distinguish three cases:

1. $M^2 < Q^2$: *NAKED SINGULARITY*

This possibility is ruled out by the principle of cosmic censorship which assumes that a gravitational collapse cannot lead to a naked singularity.

2. $M^2 > Q^2$: *TWO REAL SEPARATE SOLUTIONS*

They are situated respectively at:

$$r_- = M - \sqrt{M^2 - Q^2} \quad (\text{D.1.12})$$

$$r_+ = M + \sqrt{M^2 - Q^2}. \quad (\text{D.1.13})$$

There are two event horizons, the first inside the second one. The only peculiar feature of the resulting geometry is that in the region between r_- and r_+ you are not allowed to reverse your motion: if your direction is $r_+ \longrightarrow r_-$, you must keep it until you reaches r_- and vice versa.

3. $M^2 = Q^2$: *TWO COINCIDENT SOLUTIONS*

This is called the *extreme* Reissner-Nordström black hole. Its event horizon is situated at:

$$r_- = r_+ = M \quad (\text{D.1.14})$$

and the metric (D.1.8) in this case reduces to:

$$ds^2 = -\left(1 - \frac{M}{r}\right)^2 dt^2 + \left(1 - \frac{M}{r}\right)^{-2} dr^2 + r^2 d\Omega^2. \quad (\text{D.1.15})$$

D.2 Killing Vectors

In this section we make a brief introduction about the Killing Vectors and their meaning in General Relativity. Consider a Riemannian (or Lorentzian) manifold \mathcal{M} and a vector field $X = X^\mu \partial_\mu$ on it. If the following transformation of coordinates:

$$x^\mu \longrightarrow y^\mu = x^\mu + \epsilon X^\mu \quad (\text{D.2.1})$$

is an *isometry* for the metric g (i.e. g is invariant under this change), then X is called a *Killing Vector* of the metric at hand. Let us work out an explicit condition for Killing Vectors. Under a general change of coordinates the metric coefficients $g_{\mu\nu}$ change according to:

$$g'_{\mu\nu}(y) = \frac{\partial x^\alpha}{\partial y^\mu} \frac{\partial x^\beta}{\partial y^\nu} g_{\alpha\beta}(x), \quad (\text{D.2.2})$$

in such a way that the isometry condition becomes:

$$g'_{\mu\nu}(y) = g_{\mu\nu}(y) = \frac{\partial x^\alpha}{\partial y^\mu} \frac{\partial x^\beta}{\partial y^\nu} g_{\alpha\beta}(x). \quad (\text{D.2.3})$$

If we use Eq.(D.2.1) in Eq.(D.2.3), after few steps we arrive at:

$$\partial_\mu g_{\alpha\beta}(x) X^\mu + g_{\mu\beta}(x) \partial_\alpha X^\mu + g_{\alpha\nu}(x) \partial_\beta X^\nu = 0. \quad (\text{D.2.4})$$

We can rewrite the previous formula in two equivalent forms:

$$\mathcal{L}_X g = 0 \quad (\text{D.2.5})$$

$$\nabla_\mu X_\nu - \nabla_\nu X_\mu = 0, \quad (\text{D.2.6})$$

where \mathcal{L}_X is the *Lie derivative* along X and ∇_μ is the usual covariant derivative introduced in Eq.(D.1.6).

There are some properties satisfied by Killing Vectors:

1. If ξ_1 and ξ_2 are Killing vectors, also $\alpha\xi_1 + \beta\xi_2$ and $[\xi_1, \xi_2]$ (where the commutator is the *Lie bracket*) are Killing vectors.
2. One can show that an N -dimensional Riemannian manifold can have at most $\frac{N(N+1)}{2}$ independent Killing vectors (when this happens the manifold is called *maximally symmetric*). For example the 4-dim. Minkowski space-time admits 10 Killing vectors, which correspond to the generators of the Poincaré group.
3. If $\xi = \xi^\mu \partial_\mu$ is a Killing vector, then the product:

$$C(\lambda) := \xi^\mu \frac{dx_\mu(\lambda)}{d\lambda} \equiv \xi^\mu \dot{x}_\mu(\lambda) \quad (\text{D.2.7})$$

is constant along a geodesic $x_\mu(\lambda)$. In fact we have:

$$\begin{aligned} \frac{DC(\lambda)}{D\lambda} &:= \dot{x}^\nu(\lambda) \nabla_\nu C(\lambda) = \dot{x}^\nu (\dot{x}^\mu \nabla_\nu \xi_\mu + \xi_\mu \nabla_\nu \dot{x}^\mu(\lambda)) = \\ &= \dot{x}^\nu \dot{x}^\mu \nabla_\nu \xi_\mu + \xi_\mu \dot{x}^\nu \nabla_\nu \dot{x}^\mu = 0, \end{aligned} \quad (\text{D.2.8})$$

where in the last step the first term vanishes because of (D.2.6) and the second is zero because $x(\lambda)$ is a geodesic and therefore satisfies:

$$\frac{D\dot{x}^\mu}{D\lambda} = \dot{x}^\nu \nabla_\nu \dot{x}^\mu = 0. \quad (\text{D.2.9})$$

4. Consider the action of a point particle in a gravitational field:

$$S = \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}; \quad (\text{D.2.10})$$

since $p_\alpha = \frac{\dot{x}^\alpha}{\sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}}$ and $g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$ is constant along a geodesic, property 3 implies that $p_\alpha \xi^\alpha$ is conserved along a geodesic. This law of conservation is a generalization of what is well known in Classical Mechanics. In fact, if $g_{\mu\nu}(x)$ is independent of a coordinate x^τ , then $\mathcal{L}_\zeta g = 0$ where $\zeta = \frac{\partial}{\partial x^\tau}$, which means that ζ is a Killing vector. According to property 3, we have that $p_\mu \zeta^\mu = p_\tau$ is constant along a geodesic, and so we conclude that the momentum conjugate to the cyclic coordinate x^τ is conserved.

D.3 De Sitter and Anti-De Sitter Space-Time

In the previous Section we have defined what a Killing vector is and when a manifold is called *maximally symmetric*. In general one can show that a maximally symmetric manifold is characterized by the following equation:

$$R_{\alpha\beta\gamma\delta} = \frac{1}{N(N-1)} R (g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma}) \quad (\text{D.3.1})$$

where $R_{\alpha\beta\gamma\delta}$ is the curvature tensor and R is the Ricci scalar. Moreover, we can prove that from Eq.(D.3.1) together with the Bianchi identities

$$\nabla_\alpha \left(R^\alpha_\gamma - \frac{1}{2} \delta^\alpha_\gamma R \right) = 0 \quad (\text{D.3.2})$$

it follows that $R = \text{const.}$ In fact from Eq.(D.3.1) we easily get:

$$R_{\alpha\gamma} = \frac{1}{N} R g_{\alpha\gamma}; \quad (\text{D.3.3})$$

and if we insert (D.3.3) in (D.3.2) we get:

$$\left(\frac{1}{N} - \frac{1}{2}\right) \partial_\gamma R = 0. \quad (\text{D.3.4})$$

The previous equation tells us that if $N > 2$ (even if it can be shown — in a more elaborate way — that the same result holds also for $N = 2$) we have that R is constant. Thus the maximally symmetric spaces are uniquely specified by a curvature constant R and by the number of positive (or negative) eigenvalues of the metric $g_{\alpha\beta}$. Now let us focus on the case $N = 4$ and consider again Eq.(D.3.3) which becomes:

$$R_{\alpha\gamma} = \frac{1}{4} R g_{\alpha\gamma}, \quad (\text{D.3.5})$$

from which we obtain:

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = -\frac{1}{4} g_{\alpha\beta} R. \quad (\text{D.3.6})$$

The space with $R = 0$ is the Minkowski space-time, the space with $R > 0$ is called *De Sitter* space-time and that with $R < 0$ is called *Anti-De Sitter* space-time.

D.3.1 De Sitter (dS) space-time

We can visualize this space as the following hyperboloid in \mathbb{R}^5 :

$$-v^2 + w^2 + x^2 + y^2 + z^2 = \alpha^2 \quad (\text{D.3.7})$$

with metric:

$$ds^2 = -dv^2 + dw^2 + dx^2 + dy^2 + dz^2. \quad (\text{D.3.8})$$

We can deduce from (D.3.8) that the topology of this space is $\mathbb{R} \times \mathbb{S}^3$ (\mathbb{R} is parameterized by v and \mathbb{S}^3 by w, x, y, z) and we can introduce a new set of coordinates (t, χ, θ, ϕ) defined by the following relations:

$$v := \alpha \sinh(\alpha^{-1} t); \quad w := \alpha \cosh(\alpha^{-1} t) \cos \chi; \quad (\text{D.3.9})$$

$$x := \alpha \cosh(\alpha^{-1} t) \sin \chi \cos \theta; \quad y := \alpha \cosh(\alpha^{-1} t) \sin \chi \sin \theta \cos \phi; \quad (\text{D.3.10})$$

$$z := \alpha \cosh(\alpha^{-1} t) \sin \chi \sin \theta \sin \phi. \quad (\text{D.3.11})$$

With this choice of variables we obtain that the metric (D.3.8) takes the form:

$$ds^2 = -dt^2 + \alpha^2 \cosh^2(\alpha^{-1} t) [d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2)]. \quad (\text{D.3.12})$$

D.3.2 Anti-De Sitter (AdS) space-time

The Anti-De Sitter space AdS_4 can be represented as the following hyperboloid

$$-u^2 - v^2 + x^2 + y^2 + z^2 = 1 \quad (\text{D.3.13})$$

embedded in \mathbb{R}^5 with the following metric:

$$ds^2 = -du^2 - dv^2 + dx^2 + dy^2 + dz^2. \quad (\text{D.3.14})$$

As we did in the previous case, we can introduce the variables (t, χ, θ, ϕ) defined as in (D.3.9)-(D.3.11) and we see that the metric (D.3.14) becomes

$$ds^2 = -dt^2 + \cos^2 t [d\chi^2 + \sinh^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2)]. \quad (\text{D.3.15})$$

The topology of the metric (D.3.14) is given by $\mathbb{S}^1 \times \mathbb{R}^3$ (\mathbb{S}^1 is parameterized by (u, v) while \mathbb{R}^3 by (x, y, z)) and the symmetry group is $SO(3, 2)$, which leaves invariant the LHS of (D.3.13). In Appendix C.1 we have seen that $SO(3, 2) \cong Conf(3)$ and by the same strategy it is possible to show that $SO(2, 1) \cong Conf(1)$, where $SO(2, 1)$ is the symmetry group of the AdS_2 space.

Appendix E

Superspace Formulation: Mathematical Details

E.1 Details of the derivation of Eqs.(5.6.1)-(5.6.4)

In this Appendix we are going to show the detailed calculations leading to Eqs.(5.6.1)-(5.6.4). The reader may have noticed the similarity between the charge Q_H (1.3.8) and the Q_D, Q_K of Eqs.(5.4.6)-(5.4.9). We say “similarity” because all of them are made of two pieces, the first is the Q_{BRS} for all of them. It is easy to show that also the second pieces can be put in a similar form. Like for Q_H the second piece had the form $N_H = c^a \partial_a H$, it is easy to show that also Q_D , and Q_K can be put in the form:

$$Q_D = Q_{BRS} - 2\gamma N_D; \quad Q_K = Q_{BRS} - \alpha N_K \quad (\text{E.1.1})$$

where N_D and N_K are respectively:

$$N_D = c^a \partial_a D_0; \quad N_K = c^a \partial_a K_0 \quad (\text{E.1.2})$$

with the D_0 and K_0 given¹ by Eqs.(5.1.16)(5.1.17). So all the three operators (N_H, N_D, N_K) could be put in the general form:

$$N_X = c^a \partial_a X \quad (\text{E.1.3})$$

where X is either H, D_0 or K_0 . In the case of D_0 and K_0 , X is quadratic in the variables φ^a :

$$X = \frac{1}{2} X_{ab} \varphi^a \varphi^b \quad (\text{E.1.4})$$

¹Actually we take the classical version of (5.1.16) as we are doing Classical Mechanics.

where X_{ab} is a constant 2×2 matrix.

In order to find \widehat{N}_x (that is the superspace version of N_x) we should use Eq.(1.4.2) where Q is now our operator N_x . From the expression of N_x we get for $\delta\Phi^a(t, \theta, \bar{\theta})$ of Eq.(1.4.2):

$$\delta\Phi^a(t, \theta, \bar{\theta}) = \bar{\theta}\omega^{ab}(\bar{\varepsilon}\partial_b X) + i\bar{\theta}\theta\omega^{ab}(i\bar{\varepsilon}c^d\partial_d\partial_b X) \quad (\text{E.1.5})$$

where $\bar{\varepsilon}$ is the anticommuting parameter associated to the transformation. Given the form of X (see Eq.(E.1.4) above), we get for (E.1.5):

$$\delta\Phi^a(t, \theta, \bar{\theta}) = \bar{\theta}\bar{\varepsilon}\omega^{ab}X_{bd}[\phi^d + \theta c^d]. \quad (\text{E.1.6})$$

Note that, using superfields, the above expression can be written as:

$$\delta\Phi^a(t, \theta, \bar{\theta}) = -\bar{\varepsilon}\omega^{ab}X_{bd}\bar{\theta}\Phi^d(t, \theta, \bar{\theta}). \quad (\text{E.1.7})$$

So we obtain from Eq.(5.5.12) that the superspace expression of N_x is

$$(\mathcal{N}_x)_d^a = \omega^{ab}X_{bd}\bar{\theta}. \quad (\text{E.1.8})$$

The same kind of analysis we have done here for the Q_D and Q_K can be done also for the \overline{Q}_D and \overline{Q}_K . They can be written as:

$$\overline{Q}_D = \overline{Q}_{BRS} + 2\gamma \overline{N}_D; \quad \overline{Q}_K = \overline{Q}_{BRS} + \alpha \overline{N}_K; \quad (\text{E.1.9})$$

with

$$\overline{N}_D = \bar{c}_a\omega^{ab}\partial_b D; \quad \overline{N}_K = \bar{c}_a\omega^{ab}\partial_b K; \quad (\text{E.1.10})$$

and the superspace representation of the \overline{N}_x turns out to be:

$$(\overline{\mathcal{N}}_x)_b^a = \omega^{ac}X_{cb}\theta. \quad (\text{E.1.11})$$

Remembering the form of the D_0 and K_0 functions in their classical version (see Eqs.(5.1.16)(5.1.17)) and comparing it with the general form of X of Eq.(E.1.4) above, we get from Eqs.(E.1.2)(E.1.3) that the matrices X_{ab} associated to D and K are² exactly those of Eq.(5.6.5). This is precisely what we wanted to prove.

E.2 Representation of H,D,K in superspace

In this Appendix we will reproduce the calculations which provide the superspace representations of the operators (H, D, K) contained in **TABLE 9**. We will start

²We will call this form of X_{ab} as D_{ab} , and the one associated to K_0 as K_{ab} , to stick to the conventions of Eqs.(5.6.1)-(5.6.4).

first with the operators at time $t = 0$ which are listed in Eqs.(5.1.15)-(5.1.17). Using Eq.(1.4.2) let us first do the variations $\delta_{(H,D,K)}\Phi^a$. As (H, D_0, K_0) contain only (φ^a) their action will affect only the λ_a field contained in the superfield Φ :

$$\delta_H \lambda_q = \varepsilon[H, \lambda_q] = -i\varepsilon \frac{g}{q^3} \quad (\text{E.2.1})$$

$$\delta_H \lambda_p = \varepsilon[H, \lambda_p] = i\varepsilon p \quad (\text{E.2.2})$$

$$\delta_{D_0} \lambda_q = \varepsilon[D_0, \lambda_q] = -\frac{i}{2}\varepsilon p \quad (\text{E.2.3})$$

$$\delta_{D_0} \lambda_p = \varepsilon[D_0, \lambda_p] = -\frac{i}{2}\varepsilon q \quad (\text{E.2.4})$$

$$\delta_{K_0} \lambda_q = \varepsilon[K_0, \lambda_q] = i\varepsilon q \quad (\text{E.2.5})$$

$$\delta_{K_0} \lambda_p = \varepsilon[K_0, \lambda_p] = 0. \quad (\text{E.2.6})$$

Considering that the two superfields are:

$$\Phi^q = q + \theta \ c^q + \bar{\theta} \bar{c}_p + i\bar{\theta}\theta \lambda_p; \quad (\text{E.2.7})$$

$$\Phi^p = p + \theta \ c^p - \bar{\theta} \bar{c}_q - i\bar{\theta}\theta \lambda_q; \quad (\text{E.2.8})$$

it is very easy to see that the \mathcal{O} -operators in the RHS of Eqs.(1.4.2) can only be the following:

$$\mathcal{H} = \bar{\theta}\theta \frac{\partial}{\partial t}; \quad (\text{E.2.9})$$

$$\mathcal{D}_0 = -\frac{1}{2}\bar{\theta}\theta \sigma_3; \quad (\text{E.2.10})$$

$$\mathcal{K}_0 = -\bar{\theta}\theta \sigma_- . \quad (\text{E.2.11})$$

Next we should pass to the representation of the time-dependent operators which are related to the time-independent ones by Eqs.(5.6.6)-(5.6.8). Also for the superspace representation there will be the same relations between the two set of operators, that means:

$$\mathcal{H} = \mathcal{H}_0; \quad (\text{E.2.12})$$

$$\mathcal{D} = t\mathcal{H} + \mathcal{D}_0; \quad (\text{E.2.13})$$

$$\mathcal{K} = t^2\mathcal{H} + 2t\mathcal{D}_0 + \mathcal{K}_0. \quad (\text{E.2.14})$$

Using the above relations and the expressions obtained in Eqs.(E.2.9)-(E.2.11), it is easy to reproduce the last three operators contained in **TABLE 9**.

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